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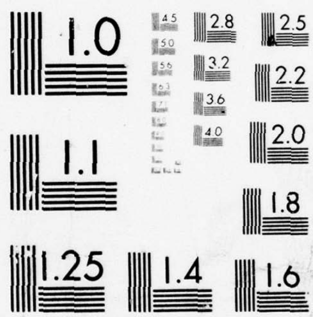

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FINITE ELEMENTS AND VARIATIONAL INEQUALITIES

R. Glowinski

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FINITE ELEMENTS AND VARIATIONAL INEQUALITIES

R. Glowinski*

Technical Summary Report #1885

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ABSTRACT

In this report we consider the various problems related to the numerical analysis of elliptic variational inequalities. After proving some abstract results concerning the approximation of variational inequalities, we consider the approximation of some specific examples by conforming and non conforming finite element methods. Various iterative methods of solution are also described.

AMS (MOS) Subject Classifications: 35J20, 35J25, 35N30.

Key Words: Numerical analysis; Finite element methods; Elliptic variational inequalities.

Work Unit Number 7 - Numerical Analysis

This report follows the text of a lecture given at the University of Wisconsin-Madison during the 1978 Spring SIAM Meeting, May 24-26.

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SIGNIFICANCE AND EXPLANATION

Many boundary value problems for elliptic equations can be formulated as variational problems in which a quadratic functional must be minimized on the subspace of functions which satisfy the boundary conditions. The best known example of such variational problems arises when Laplace's equation must be solved subject to prescribed boundary values, in which case the variational problem requires the minimization of the Dirichlet integral. This provides the starting point for the widely-used finite element method for computing solutions of potential problems such as occur in steady state heat conduction or inviscid fluid flow, for example.

An elliptic variational inequality is a generalization of variational problems of the above type: it is required to minimize a quadratic functional on a convex set. Elliptic variational inequalities arise in contexts in which a physical system is subject to restraints, as when a membrane is stretched over an obstacle. (If no obstacle is present, the deflection of the membrane is governed by Laplace's equation. The obstacle is the new feature that leads to the variational inequality. The region of contact between the membrane and the obstacle is not known beforehand.)

This report first describes the theory of approximating elliptic variational inequalities, and then goes on to discuss the numerical solution of such problems using finite elements.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CONTENTS

FINITE ELEMENTS AND VARIATIONAL INEQUALITIES

R. GLOWINSKI*

INTRODUCTION

During these last years many works concerned with the Numerical Analysis of Variational Inequality problems have been published. We would like in this paper to describe some of the results which have been obtained with emphasis on the Finite Element Approximation of these Inequalities.

The content is as follows :

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1. ABSTRACT ELLIPTIC VARIATIONAL INEQUALITIES. EXISTENCE, UNIQUENESS, APPROXIMATION.

1.1. Orientation.

In Sec. 1 we just consider, following GLOWINSKI [53, Ch. 1,2,3] some simple classes of Elliptic Variational Inequalities (E.V.I.) and their approximation by Galerkin type methods. More specific examples will be considered in Sec. 2.

1.2. Functional context.

We introduce the following mathematical objects :

- A real Hilbert space V equipped with the inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$.
- A bilinear continuous form $a : V \times V \to \mathbb{R}$, V-elliptic (i.e. $\exists \alpha > 0$ such that $a(v,v) \geq \alpha \|v\|^2 \ \forall v \in V$) ; we don't assume that $a(\cdot,\cdot)$ is symmetric.
- A form $L : V \to \mathbb{R}$, linear and continuous.
- A closed, convex, non empty subset K of V.
- A functional $j : V \to \bar{\mathbb{R}}$ $(= \mathbb{R} \cup \{+\infty\} \cup \{-\infty\})$, convex, lower semi continuous (l.s.c.), proper (i.e. $j(v) > -\infty \ \forall v \in V$, $j \not\equiv +\infty$).

1.3. Two classes of E.V.I.

Let us consider now

$$(EVI)_1 \quad \begin{cases} \text{Find } u \in K \text{ such that} \\ a(u,v-u) \geq L(v-u) \ \forall v \in K, \end{cases}$$

and

$$(EVI)_2 \quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v-u)+j(v)-j(u) \geq L(v-u) \quad \forall v \in V. \end{cases}$$

In the sequel $(EVI)_1$ (resp $(EVI)_2$) will be denoted as an EVI of the first (resp. second) kind.

Remark 1.1. : We can find in the litterature more complicated EVI, (cf. for example LIONS [65]) and also the generalization called Quasi Variational Inequalities (QVI) introduced by BENSOUSSAN-LIONS (see for example BENSOUSSAN-LIONS [9], BAIOCCHI-CAPELO [6] and the bibliography therein).

Remark 1.2 : If $K=V$ in $(EVI)_1$ and $j \equiv 0$ in $(EVI)_2$, then both problems reduce to the standard linear variational equation

$$(1.1) \quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = L(v) \ \forall v \in V. \end{cases}$$

Remark 1.3 : The distinction between $(EVI)_1$ and $(EVI)_2$ is rather artificial (theoretically at least) since $(EVI)_1$ is equivalent to

$$(1.2) \quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v-u) + I_K(v) - I_K(u) \geq L(v-u) \ \forall v \in V \end{cases}$$

where I_K is defined by

$$I_K(v) = \begin{cases} 0 \text{ if } v \in K, \\ +\infty \text{ if } v \notin K. \end{cases}$$

The functional I_K is called the indicator functional of K, and since K is a closed, convex, non empty subset of V, I_K is l.s.c., convex, proper. Therefore $(EVI)_1$ is a special case of $(EVI)_2$; however formulation $(EVI)_1$ is usually more practical.

Remark 1.4 : If $a(\cdot,\cdot)$ is symmetric $(EVI)_1$ and $(EVI)_2$ are respectively equivalent to the minimization problems

$$(\pi_1) \quad \begin{cases} \text{Find } u \in K \text{ such that} \\ J(u) \leq J(v) \quad \forall v \in K \end{cases}$$

where

$$(1.3) \qquad J(v) = \frac{1}{2} a(v,u) - L(v),$$

and

$$(\pi_2) \quad \begin{cases} \text{Find } u \in K \text{ such that} \\ J(u) + j(u) \leq J(v) + j(v) \end{cases}$$

where $J(\cdot)$ is still defined by (1.3).

1.4. Existence and Uniqueness results for $(EVI)_1$, $(EVI)_2$.

From LIONS-STAMPACCHIA [66] we have the following

Theorem 1.1 : If the above hypotheses on V,a,L,K hold then $(EVI)_1$ has a unique solution.

Proof : (1) Uniqueness

Let u_1 and u_2 be two solutions. Then

$$(1.4) \qquad a(u_1,v-u_1) \geq L(v-u_1) \quad \forall v \in K, \; u_1 \in K,$$

$$(1.5) \qquad a(u_2,v-u_2) \geq L(v-u_2) \quad \forall v \in K, \; u_2 \in K.$$

Taking $v=u_2$ in (1.4), $v=u_1$ in (1.5) we obtain by addition and using the V-ellipticity of $a(\cdot,\cdot)$ that

$$\alpha \|u_2-u_1\|^2 \leq a(u_2-u_1,u_2-u_1) \leq 0$$

which implies the uniqueness.

(2) Existence

It is known from the Riesz representation theorem that there exists $A \in \mathcal{L}(V,V)$ and $\ell \in V$ such that

$$(1.6) \qquad a(u,v) = (Au,v) \quad \forall u,v \in V,$$

$$(1.7) \qquad L(v) = (\ell,v) \quad \forall v \in V.$$

Then if u is solution of $(EVI)_1$ we have

$$(1.8) \qquad \begin{cases} (Au,v-u) \geq (\ell,v-u) \quad \forall v \in K, \\ u \in K \end{cases}$$

which is equivalent to $(EVI)_1$. Then (1.8) is equivalent to

$$(1.9)\quad \begin{cases} (u-\rho(Au-\ell)-u, v-u) \le 0 \quad \forall v \in K, \\ u \in K,\ \rho > 0\ , \end{cases}$$

and (1.9) is equivalent to

$$(1.10)\qquad u = P_K(u-\rho(Au-\ell)\ ,\ \rho > 0\ ,$$

where in (1.10) P_K is the projection operator from V to K in the $\|\cdot\|$-norm.

It follows from (1.10) that every solution of $(EVI)_1$ is also solution of the fixed-point problem (1.10) for any $\rho > 0$, and conversely if there exists a particular ρ such that (1.10) has a solution, this solution is also solution of $(EVI)_1$.

A sufficient condition for (1.10) to have a solution is that the mapping from V to V

$$v \to P_K(v-\rho(Av-\ell))$$

is strictly and uniformly contracting for ρ well chosen.
Let denote by w the vector $P_K(v-\rho(Av-\ell))$ and let consider

$$w_i = P_K(v_i-\rho(Av_i-\ell))\ ,\ i=1,2\ .$$

Since the projection mapping P_K is a contraction, we have

$$(1.11)\qquad \|w_2-w_1\| \le \|v_2-v_1-\rho A(v_2-v_1)\|\ .$$

From (1.6), (1.11) we obtain

$$\begin{cases} \|w_2-w_1\|^2 \le \|v_2-v_1\|^2 -2\ a(v_2-v_1, v_2-v_1)+\rho^2\ \|A(v_2-v_1)\|^2 \le \\ \qquad\qquad \le \|v_2-v_1\|^2 -2\rho\alpha\ \|v_2-v_1\|^2 +\rho^2\ \|A\|^2\ \|v_2-v_1\|^2 \end{cases}$$

so that

$$(1.12)\qquad \|w_2-w_1\|^2 \le (1-2\rho\alpha + \rho^2\ \|A\|^2)\ \|v_2-v_1\|^2\ .$$

From (1.12), the above mapping will be strictly and uniformly contracting if

$$(1.13) \qquad 0 < \rho < \frac{2\alpha}{\|A\|^2} \quad .$$

If ρ obeys (1.13), the fixed point problem (1.10) has a solution which is also the solution of $(EVI)_1$ and we know that this solution is unique.

Remark 1.5 : The proof of Theorem 1.1 suggests the following algorithm for solving $(EVI)_1$:

$$(1.14) \qquad u^o \in V, \text{ given}$$

then for $n \geq 0$

$$(1.15) \qquad u^{n+1} = P_K(u^n - \rho(Au^n - \ell)).$$

From the proof of Theorem 1.1 it follows that for $0 < \rho < \frac{2\alpha}{\|A\|^2}$ the sequence $(u^n)_n$ defined by (1.14),(1.15) converges strongly in V to the solution u of $(EVI)_1$.

Practically the interest in (1.14),(1.15) is quite limited in most applications (at least in the above form) since we usually don't know ℓ or A, and to project on K is in most cases a very complicated operation. We should observe that if $a(\cdot,\cdot)$ is symmetric, then A is also symmetric and (1.14),(1.15) is a gradient with projection algorithm ; cf., e.g., CEA [28] for a study of these methods. ∎

Concerning $(EVI)_2$ it follows from LIONS-STAMPACCHIA, loc.cit., that

Theorem 1.2 : *If the above hypotheses on* V,a,L,j *hold then* $(EVI)_2$ *has a unique solution.*

We refer for the proof to LIONS-STAMPACCHIA, loc. cit., and also to GLOWINSKI [53] ; in fact in this proof which is a variant of the proof of Theorem 1.1, one still uses a fixed point technique.

1.5. Internal Approximation of $(EVI)_1$

The assumptions on V,a,K,L are those of Sec. 1.2.

1.5.1. Approximation of V and K.

The parameter h converging to zero, the space V is approximated by the family $(V_h)_h$ where the V_h are closed subspaces of V (usually the V_h are finite-dimensional).

Then the convex set K is approximated by $(K_h)_h$ where the K_h are closed convex subsets of V_h ; we do not assume that $K_h \subset K$. We do assume however that $(K_h)_h$ has the two following properties

(i) If $(v_h)_h$ is such that $v_h \in K_h \quad \forall h$ then all the weak cluster points of $(v_h)_h$ belong to K,

(ii) There exist $\chi, \overline{\chi} = K$ and $r_h : \chi \to K_h$ such that

$$\lim_{h\to 0} r_h v = v \text{ strongly in } V, \forall v \in \chi \quad .$$

Remark 1.6 : If $K_h \subset K \quad \forall h$, then (i) is automatically satisfied.

1.5.2. Approximation of $(EVI)_1$.

We approximate $(EVI)_1$ by

$$(EVI)_{1h} \begin{cases} \text{Find } u_h \in K_h \text{ such that} \\ a(u_h, v_h - u_h) \geq L(v_h - u_h) \quad \forall v_h \in K_h. \end{cases}$$

Remark 1.7 : In most cases it will be necessary to approximate also $a(\cdot,\cdot)$ and $L(\cdot)$ by $a_h(\cdot,\cdot)$ and $L_h(\cdot)$ (usually defined - in practical cases - from $a(\cdot,\cdot)$ and $L(\cdot)$ by a numerical integration procedure). Since there is nothing new on that point compared to the classical linear case we shall not insist more about this problem for which we refer to CIARLET [33].

It is easily proved that

Proposition 1.1 : Problem $(EVI)_{1h}$ has a unique solution.

1.5.3. Convergence results.

Let us prove now the following convergence theorem

Theorem 1.3 : If the above hypotheses on $(V_h)_h$, $(K_h)_h$ hold then

$$\lim_{h\to 0} \|u_h - u\| = 0 \tag{1.16}$$

where u, u_h are respectively solutions of $(EVI)_1$, $(EVI)_{1h}$.

Proof : (1) Estimates for u_h

We have from $(EVI)_{1h}$

$$a(u_h,u_h) \le a(u_h,v_h) - L(v_h - u_h) \quad \forall\, v_h \in K_h$$

which implies $\forall\, v_h \in K_h$ that

$$\alpha \|u_h\|^2 \le \|A\|\,\|u_h\|\,\|v_h\| + \|L\|_* \|v_h\| + \|L\|_* \|u_h\| . \tag{1.17}$$

From (ii) in Sec. 1.5.1. and (1.17) we obtain

$$\alpha \|u_h\|^2 \le \|A\|\,\|u_h\|\,\|r_h v\| + \|L\|_* \|r_h v\| + \|L\|_* \|u_h\| \quad \forall\, v \in \chi .$$

Now take $v_o \in \chi$. Then with C_i denoting various constants depending on v_o but not on h, we have from (ii)

$$\|r_h v_o\| \le C_o \quad \forall\, h$$

which implies

$$\alpha \|u_h\|^2 \le C_1 \|u_h\| + C_2 \quad \forall h . \tag{1.18}$$

In turn (1.18) implies the boundedness of $(u_h)_h$ in V.

(2) Weak convergence of $(u_h)_h$.

We can extract from $(u_h)_h$ a subsequence, still denoted by $(u_h)_h$, such that

$$u_h \to u^* \text{ weakly in V.} \tag{1.19}$$

From (i) in Sec. 1.5.1. we have

(1.20) $\quad u^* \in K.$

From (1.19) and property (ii) we obtain taking the limit in

$$a(u_h,u_h) \leq a(u_h,r_h v)-L(r_h v-u_h) \quad \forall v \in \chi$$

that

(1.21) $$\liminf_{h\to 0} a(u_h,u_h) \leq a(u^*,v)-L(v-u^*) \quad \forall v \in \chi .$$

Then we observe that

$$\alpha \|u_h-u^*\|^2 \leq a(u_h-u^*,u_h-u^*) = a(u_h,u_h)+a(u^*,u^*)-a(u_h,u^*)-a(u^*,u_h)$$

implies in the limit

(1.22) $$\liminf_{h \to 0} a(u_h,u_h) \geq a(u^*,u^*)$$

which is a (well-known) weak lower semi continuity property.

From (1.21),(1.22) it follows that

(1.23) $$a(u^*,u^*) \leq a(u^*,v)-L(v-u^*) \quad \forall v \in \chi .$$

Since $\overline{\chi}=K$, (1.23) also holds $\forall v \in K$, so that with (1.20) we have

$$\begin{cases} a(u^*,v-u^*) \geq L(v-u^*) & \forall v \in K, \\ u^* \in K. \end{cases}$$

Thus u^* is a solution of $(EVI)_1$. But from the uniqueness of such a solution we have $u^* = u$. The uniqueness property implies also that the whole sequence $(u_h)_h$ converges weakly to u.

(3) Strong convergence of $(u_h)_h$.

We have

$$\begin{cases} \alpha \|u_h-u\|^2 \le a(u_h-u,u_h-u) = a(u_h,u_h)+a(u,u)-a(u_h,u)-a(u,u_h) \le \\ \qquad \le a(u_h,r_hv)-L(r_hv-u_h)+a(u,u)-a(u_h,u)-a(u,u_h) \quad \forall v \in \chi . \end{cases}$$

From the above relation, from property (ii) and from the weak convergence of $(u_h)_h$, we have in the limit

$$\alpha \limsup \|u_h-u\|^2 \le a(u,v)-L(v-u)-a(u,u) = a(u,v-u)-L(v-u) \ \forall v \in \chi .$$

But since $\overline{\chi}$=K we also have

$$(1.24) \qquad \alpha \limsup \|u_h-u\|^2 \le a(u,v-u)-L(v-u) \quad \forall v \in K.$$

Taking v=u in (1.24) it follows that

$$\alpha \limsup \|u_h-u\|^2 \le 0$$

which implies that $\lim_{h\to 0} \|u_h-u\| = 0$ i.e. the strong convergence.

Remark 1.8 : Error estimates for some EVI's of the first type have been obtained by several authors (see Sec. 2 for more details). But like in many non-linear problems, the methods used to obtain these estimates are specific to the particular problem under consideration.

This remark still holds for the approximation of EVI's of the second kind which is the subject of the next sub-Section 1.6.

1.6. Internal Approximation of $(EVI)_2$.

The assumptions on V, a(•,•), L(•), j(•) are those of Sec. 1.2. Furthermore we assume for simplicity that

(1.25) j(.) is continuous over V.

(In Sec. 5 an important family of $(EVI)_2$'s for which j(•) is non continuous will be considered also).

1.6.1. Approximation of V.

The space V is approximated by the family $(V_h)_h$, V_h being a closed subspace of $V(\dim V_h < +\infty$ in applications). We assume that $(V_h)_h$ has the following property

(i) There exist $\mathcal{V} \subset V, \overline{\mathcal{V}} = V$ and $r_h : \mathcal{V} \to V_h$ such that

$$\lim_{h \to 0} r_h v = v \text{ strongly in } V, \forall v \in \mathcal{V} .$$

1.6.2. Approximation of $j(\cdot)$.

The functional $j(\cdot)$ is approximated by $(j_h)_h$ where

$$(1.26) \quad \begin{cases} j_h : V \to \overline{\mathbb{R}}, \\ j_h \text{ is convex, l.s.c., uniformly proper in } h ; \end{cases}$$

the last property implies the existence of $\lambda : V \to \mathbb{R}$, linear and continuous and of $\mu \in \mathbb{R}$ such that

$$(1.27) \qquad j_h(v_h) \geq \lambda(v_h) + \mu \quad \forall v_h \in V_h, \forall h .$$

We shall assume also that $(j_h)_h$ obeys

(ii) If $v_h \to v$ weakly in V then

$$\liminf j_h(v_h) \geq j(v)$$

(iii) $$\lim_{h \to 0} j_h(r_h v) = j(v) \quad \forall v \in \mathcal{V} .$$

Remark 1.9 : In all the applications we know, if $j(\cdot)$ is continuous, then it is always possible to construct continuous $j_h(\cdot)$ obeying (ii), (iii).

Remark 1.10 : If $j_h = j \quad \forall h$, then (1.26), (ii), (iii), are automatically satisfied.

1.6.3. Approximation of $(EVI)_2$

We approximate $(EVI)_2$ by

$$(EVI)_{2h} \quad \begin{cases} \underline{\text{Find}}\ u_h \in V_h\ \underline{\text{such that}} \\ a(u_h, v_h-u_h)+j_h(v_h)-j_h(u_h) \geq L(v_h-u_h)\ \forall v_h \in V_h\ . \end{cases}$$

It is then easily proved that

Proposition 1.2 : If the above hypotheses on $(V_h)_h$, $(j_h)_h$ hold then $(EVI)_{2h}$ has a unique solution.

Remark 1.11 : Remark 1.7 of Sec. 1.5.2. still holds for $(EVI)_{2h}$.

1.6.4. Convergence results.

Using a variant of the proof of Theorem 1.3 we obtain (see GLOWINSKI [53] for more details)

Theorem 1.4 : If the above hypotheses on $(V_h)_h$, $(j_h)_h$ hold then

$$\lim_{h\to 0} \|u_h-u\| = 0\ ,\ \lim_{h\to 0} j_h(u_h) = j(u)$$

where u, u_h are respectively solutions of $(EVI)_2$, $(EVI)_{2h}$.

2. - SPECIFIC EXAMPLES AND ERROR ESTIMATES FOR CONFORMING FINITE ELEMENT METHODS.

2.1. Stationary obstacle problems.

Problems of this type are fairly simple but provide a good mathematical model for several important applications. Futhermore obstacle problems are those for which the finite element approximation error analysis is the most achieved.

2.1.1. Formulation of a particular obstacle problem.

Let Ω be a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\Gamma = \partial\Omega$. We consider with $x = \{x_i\}_{i=1}^N$ and $\nabla = \{\frac{\partial}{\partial x_i}\}_{i=1}^N$ the particular obstacle problem

$$
(2.1)\qquad \begin{cases} \underline{\text{Find}}\ u \in K\ \underline{\text{such that}} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq \int_\Omega f(v-u)\,dx \quad \forall v \in K, \end{cases}
$$

where in (2.1), $f \in L^2(\Omega)$ and K is defined by

$$
(2.2)\qquad K = \{v \in H^1(\Omega),\ v \geq \psi \ \text{ a.e. on } \Omega,\ v|_\Gamma = g\}
$$

with ψ and g given functions defined respectively on Ω and Γ.

2.1.2. Physical interpretation.

Assume that $\Omega \subset \mathbb{R}^2$; then a classical interpretation of (2.1), (2.2) is that u represents the small vertical displacements of an elastic membrane Ω under the effects of a field of vertical forces, whose intensity is given by f (f represents a surface density of vertical forces). This membrane is fixed on its boundary Γ ($u=g$) and lies over an obstacle, whose height is given by $\psi(u \geq \psi)$; see Fig. 2.1 for a geometrical description of the phenomenon.

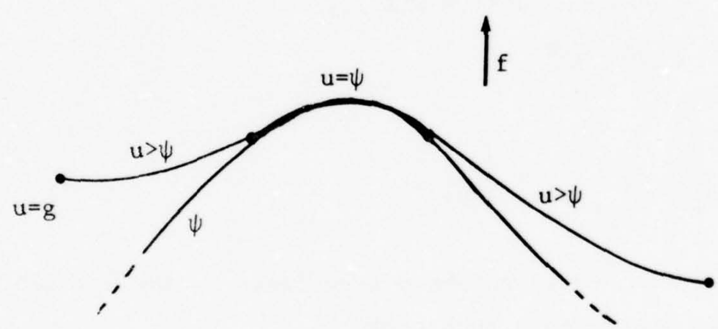

Figure 2.1

2.1.3. Other phenomena related to obstacle problems.

Similar EVI's also occur, sometimes with other type of boundary conditions and/or non symmetric bilinear forms, in mathematical models for the following problems :

- Lubrication phenomena (see, e.g., CRYER [41], [42], MARZULLI [67], GLOWINSKI-LIONS-TREMOLIERES [57, ch. 2, Sec. 5] for finite difference treatments and more references, and CAPRIZ [27] for a discussion of the mathematical modelling).

. Filtration of liquids in porous media (see in particular BAIOCCHI [1], [2], [3], COMMINCIOLI [35], BAIOCCHI-BREZZI-COMMINCIOLI [5], CRYER-FETTER [43], BAIOCCHI-CAPELO [6] and the numerous references therein).

. Two dimensional potential flows of inviscid fluids (cf. BREZIS-STAMPACCHIA [17], [18], BREZIS [11], CIAVALDINI-POGU-TOURNEMINE [34], ROUX [78] and the references therein).

. Wake problems (cf. BOURGAT-DUVAUT [10]).

This list is far from complete and we also have applications in Biomathematics, Economics, Semi-conductors, etc...

2.1.4. Interpretation of (2.1), (2.2) as a free boundary problem.

Let define from the solution u of (2.1),(2.2)

$$\Omega^+ = \{x \mid x \in \Omega,\ u(x) > \psi(x)\} \quad ,$$

$$\Omega^o = \{x \mid x \in \Omega,\ u(x) = \psi(x)\} \quad ,$$

$$\gamma = \partial\Omega^+ \cap \partial\Omega^o \quad ,$$

and then

$$u_+ = u|_{\Omega^+} \ , \ u_o = u|_{\Omega^o} \ .$$

Classically (2.1), (2.2) has been formulated as the problem of finding γ (the free boundary) and u such that

(2.3) $\quad - \Delta u = f$ on Ω^+,

(2.4) $\quad u = \psi$ on Ω^o ,

(2.5) $\quad u = g$ on Γ,

(2.6) $\quad u_+|_\gamma = u_o|_\gamma$.

The physical interpretation of (2.3)-(2.6) is the following : (2.3) means that on Ω^+ the membrane is strictly over the obstacle and has a purely elastic behaviour ; (2.4) means that on Ω^o the membrane is in contact with the obstacle ; (2.6) is a transmission relation on the free boundary.

In fact (2.3)-(2.6) are not sufficient to characterize u, therefore it is necessary to add other transmission properties ; for instance if ψ is smooth enough (let say $\psi \in H^2(\Omega)$), we should require the "continuity" of ∇u on γ (we may ask $\nabla u \in H^1(\Omega) \times H^1(\Omega)$).

Remark 2.1 : This kind of free boundary interpretation holds for the other examples considered in the sequel.

2.1.5. Existence, Uniqueness, Regularity of the solution.

Concerning the existence and uniqueness of a solution of (2.1),(2.2) we can easily prove

Theorem 2.1 : Assume that Γ is smooth and that $\psi \in H^1(\Omega)$, $g \in H^{1/2}(\Gamma)$ with

$$\psi|_\Gamma \leq g \text{ a.e. on } \Gamma$$

then (2.1),(2.2) has a unique solution.

Remark 2.2 : The above theorem holds for $f \in (H^1(\Omega))'$ and for fairly discontinuous ψ.

Concerning the regularity of u let us recall the following classical results of BREZIS-STAMPACCHIA [19] :

$$\left\{\begin{array}{l} \text{If } \Gamma \text{ is sufficiently smooth, if, for } p \in]1,+\infty[\,,\ f \in L^p(\Omega) \cap (H^1(\Omega))', \\ \psi \in W^{2,p}(\Omega)\ ,\ g = \tilde{g}|_\Gamma \text{ with } \tilde{g} \in W^{2,p}(\Omega), \text{ then } u \in W^{2,p}(\Omega). \end{array}\right.$$

Actually the above results have been refined by BREZIS [12],[13] and very sophisticated properties of the solution and of the free boundary have been obtained by Lewy-Stampacchia, Brezis, Kinderlehrer, Nirenberg, Schaeffer, etc...

2.1.6. Finite element Approximations of (2.1),(2.2). (I) Piecewise linear Approximations.

We consider in this subsection conforming finite element approximations of order one of the obstacle problem (2.1),(2.2). Piecewise quadratic approximations are considered in Sec. 2.1.7 and non conforming approximations of mixed type in Sec. 4.

We assume for simplicity that Ω is a bounded polygonal domain of $\mathbb{R}^2$. We assume also that $\psi \in H^1(\Omega) \cap C^0(\bar{\Omega})$, $g \in H^{1/2}(\Gamma) \cap C^0(\Gamma)$. We introduce then a standard triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ such that

$$\bigcup_{T \in \mathcal{T}_h} T = \bar{\Omega} ,$$

with as usual, h = length of the largest side of $\mathcal{T}_h$.

Let define now

$$\Sigma_h = \{P \in \bar{\Omega},\ P \text{ vertex of } T \in \mathcal{T}_h\}$$

$$\mathring{\Sigma}_h = \{P \in \Sigma_h\ ,\ P \notin \Gamma\} = \Sigma_h \cap \Omega.$$

We approximate then $H^1(\Omega)$ and K by respectively

(2.7) $$V_h = \{v_h \in C^0(\bar{\Omega}),\ v_h|_T \in P_1 \quad \forall T \in \mathcal{T}_h\} ,$$

(2.8) $$K_h = \{v_h \in V_h,\ v_h(P) \geq \psi(P) \quad \forall P \in \mathring{\Sigma}_h,\ v_h(P) = g(P) \quad \forall P \in \Sigma_h \cap \Gamma\}$$

where in (2.7) and the sequel (for $k \geq 0$) P_k = space of polynomials in two variables of degree $\leq k$.

Finally we approximate (2.1), (2.2) by (2.8) and

(2.9) $$\begin{cases} \text{Find } u_h \in K_h \text{ such that} \\ \displaystyle\int_\Omega \nabla u_h \cdot \nabla (v_h - u_h)\,dx \geq \int_\Omega f(v_h - u_h)\,dx \quad \forall v_h \in K_h. \end{cases}$$

Proposition 2.1 : The approximate obstacle problem (2.8), (2.9) has a unique solution.

Concerning the convergence of the approximate solutions as $h \to 0$, we refer to GLOWINSKI [53, Chapter 4, Sec. 2] for the case where u is not very smooth. Below we shall briefly consider the derivation of error estimates, in the $H^1(\Omega)$ norm, if $u, \psi, \tilde{g} \in H^2(\Omega)$. We follow very closely the analysis of BREZZI-HAGER-RAVIART [21, Sec. 4] (in which $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega uv\,dx$).

To obtain these error estimates we need the following

Lemma 2.1 : Assume that $u \in H^2(\Omega)$; then

(2.10) $\qquad -\Delta u - f \geq 0$ a.e. on Ω,

(2.11) $\qquad (-\Delta u - f)(u - \psi) = 0$ a.e. on Ω.

Proof : see BREZIS [14].

Lemma 2.2 : Let u and u_h be respectively solutions of (2.1), (2.2) and (2.8), (2.9). Then

(2.12) $$a(u_h - u, u_h - u) \leq a(u_h - u, v_h - u) + a(u, v_h - u_h) - \int_\Omega f(v_h - u_h)\,dx \quad \forall v_h \in K_h.$$

Proof : Following BREZZI-HAGER-RAVIART [21, Theorem 2.1] we have $\forall v_h \in K_h$

$$\begin{cases} a(u_h - u, u_h - u) = a(u_h - u, v_h - u) + a(u_h - u, u_h - v_h) = \\ = a(u_h - u, v_h - u) + a(u, v_h - u_h) - \int_\Omega f(v_h - u_h) + \int_\Omega f(v_h - u_h)\,dx - a(u_h, v_h - u_h). \end{cases}$$

Since v_h obeys (2.9), we have

$$\int_\Omega f(v_h - u_h)\,dx - a(u_h, v_h - u_h) \leq 0 \quad \forall v_h \in K_h$$

which combined with the above equation implies (2.12). ∎

We prove now

Theorem 2.2 : If $f \in L^2(\Omega)$, $\psi \in H^2(\Omega)$, $g = \tilde{g}|_\Gamma$, $\tilde{g} \in H^2(\Omega)$ and if the angles of $\mathcal{T}_h$ are bounded below by $\theta_o > 0$, independent of h, then

(2.13) $$\|u_h - u\|_{H^1(\Omega)} = O(h) ,$$

where u and u_h are respectively solutions of (2.1), (2.2) and (2.8), (2.9).

Proof : We follow again BREZZI-HAGER-RAVIART, loc.cit., Theorem 4.1 (see also FALK [48]). We have from Green's formula

$$(2.14)\qquad a(u,v) = \int_\Omega \nabla u\cdot\nabla v\ dx = -\int_\Omega \Delta u\ v\ dx + \int_\Gamma \frac{\partial u}{\partial n}\, v\ d\Gamma \quad \forall\, v \in H^1(\Omega).$$

Since $v_h - u_h \in H_o^1(\Omega)\ \forall v_h \in K_h$ it follows then from (2.14) that

$$(2.15)\qquad a(u,v_h-u_h) - \int_\Omega f(v_h-u_h)dx = \int_\Omega (-\Delta u - f)(v_h-u_h)dx \quad \forall\, v_h \in K_h.$$

Let π_h be the operator of V_h-interpolation on Σ_h. Then since $\Omega \subset \mathbf{R}^2$ we have $H^2(\Omega) \subset C^o(\overline{\Omega})$ and $u \in H^2(\Omega)$, $u \ge \psi$ on Ω imply

$$(2.16)\qquad \pi_h u \in K_h.$$

Taking $v_h = \pi_h u$ in (2.12),(2.15) we obtain

$$(2.17)\qquad a(u_h-u,u_h-u) \le a(u_h-u,\pi_h u-u) + \int_\Omega (-\Delta u - f)(\pi_h u - u_h)dx\ .$$

Observe that

$$(2.18)\qquad \pi_h u - u_h = (\pi_h u - u) + (\psi - \pi_h\psi) + (u-\psi) + (\pi_h\psi - u_h).$$

Let $w = -\Delta u - f$; then $w \in L^2(\Omega)$ and from (2.18)

$$(2.19)\qquad \left\{ \begin{aligned} \int_\Omega w(\pi_h u - u_h)dx = \int_\Omega w(\pi_h u - u)dx + \int_\Omega w(\psi - \pi_h\psi)dx + \int_\Omega w(u-\psi)dx\ + \\ + \int_\Omega w(\pi_h\psi - u_h)dx\ . \end{aligned} \right.$$

Since Lemma 2.1 holds we have $w \ge 0$ a.e. and $w(u-\psi) = 0$ a.e. ; moreover since $u_h \in K_h$ we have $\pi_h\psi - u_h \le 0$ on $\overline{\Omega}$. It follows then from (2.19) that

$$(2.20)\qquad \int_\Omega w(\pi_h u - u_h)dx \le ||w||_{L^2(\Omega)} \left(||\pi_h u - u||_{L^2(\Omega)} + ||\pi_h\psi - \psi||_{L^2(\Omega)} \right).$$

Since $u,\psi \in H^2(\Omega)$ we have (since the angle condition holds)

$$(2.21)\qquad ||\pi_h u - u||_{H^1(\Omega)} \le Ch\, ||u||_{H^2(\Omega)}\ , ||\pi_h u - u||_{L^2(\Omega)} \le Ch^2\, ||u||_{H^2(\Omega)}\ ,$$

(2.22) $\quad \|\pi_h\psi-\psi\|_{H^1(\Omega)} \le C\, h\|\psi\|_{H^2(\Omega)}\, , \; \|\pi_h\psi-\psi\|_{L^2(\Omega)} \le Ch^2\|\psi\|_{H^2(\Omega)}$

where the C's are independent of h,u,ψ .

It follows then from (2.17),(2.20)-(2.22) that

(2.23) $\quad |u_h-u|_{1,\Omega} = O(h)$

where

$$|v|_{1,\Omega} = \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}$$

(and where below $\|v\|_{1,\Omega} = \|v\|_{H^1(\Omega)}$).

To estimate $\|u_h-u\|_{1,\Omega}$ we observe that, since Ω is bounded

(2.24) $\quad \|v\|_{1,\Omega} \le C|v|_{1,\Omega} \quad \forall v \in H^1_0(\Omega)$, C <u>independent of</u> v.

It follows then from (2.24) and from $u_h-\pi_h u \in H^1_0(\Omega)$ that

$$(2.25)\quad \left\{\begin{aligned} \|u_h-u\|_{1,\Omega} &\le \|u_h-\pi_h u\|_{1,\Omega} + \|\pi_h u-u\|_{1,\Omega} \le \\ &\le C|u_h-\pi_h u|_{1,\Omega} + \|\pi_h u-u\|_{1,\Omega} \le \\ &\le C|u_h-u|_{1,\Omega} + C|u-\pi_h u|_{1,\Omega} + \|\pi_h u-u\|_{1,\Omega} .\end{aligned}\right.$$

Then (2.13) follows clearly from (2.21),(2.23),(2.25).

To our knowledge the first O(h) estimates for $\|u_h-u\|_{1,\Omega}$ have been obtained, for <u>piecewise linear approximations</u>, by FALK [46] and then MOSCO-STRANG [72] . These works have been followed by FALK [47], [48] (see also GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 1], CIARLET [33]). In our opinion one finds in FALK [48] the most complete analysis for piecewise linear approximations, since it also considers <u>non convex</u> and/or <u>non polygonal</u> Ω. The problem of obtaining, via a generalization of the Aubin-Nitsche trick, L^2-estimates (of optimal order) is not completely solved yet ; however for some partial results in that direction see NATTERER [76], MOSCO [70] and the references therein.

To conclude with piecewise linear approximation, let us mention that under suitable hypothesis BAIOCCHI [4] (resp. NITSCHE [77])have obtained for the obstacle problem $\|u_h-u\|_{L^\infty(\Omega)} = O(h^{2-\varepsilon})$, $\varepsilon > 0$, arbitrarily small (resp. $\|u_h-u\|_{L^\infty(\Omega)} = O(h^2|\mathrm{Log}\ h|)$).

2.1.7. Finite Element Approximations of (2.1), (2.2). (II) Piecewise quadratic approximations.

With Σ_h, $\overset{\circ}{\Sigma}_h$ as in Sec. 2.1.8 define

$$\Sigma'_h = \{P \in \bar{\Omega}\ ,\ P \text{ midpoint of a side of } T \in \mathcal{T}_h\},$$

$$\overset{\circ}{\Sigma}{}'_h = \{P \in \Sigma'_h\ ,\ P \notin \Gamma\}\ ,$$

$$\Sigma''_h = \Sigma_h \cup \Sigma'_h\ ,\ \overset{\circ}{\Sigma}{}''_h = \overset{\circ}{\Sigma}_h \cup \overset{\circ}{\Sigma}{}'_h\ .$$

We approximate $H^1(\Omega)$ and K by

(2.26) $\quad V_h = \{v_h \in C^0(\bar{\Omega})\ ,\ v_h|_T \in P_2 \quad \forall T \in \mathcal{T}_h\}$,

$(2.27)_1 \quad K^1_h = \{v_h \in V_h\ ,\ v_h(P) \geq \psi(P) \quad \forall P \in \overset{\circ}{\Sigma}{}''_h,\ v_h(P) = g(P) \quad \forall P \in \Sigma''_h \cap \Gamma\}$,

$(2.27)_2 \quad K^2_h = \{v_h \in V_h\ ,\ v_h(P) \geq \psi(P) \quad \forall P \in \overset{\circ}{\Sigma}{}'_h\ ,\ v_h(P) = g(P) \quad \forall P \in \Sigma''_h \cap \Gamma\}$.

We observe that in K^2_h the condition $v_h(P) \geq \psi(P)$ is only required on the side midpoints.

We approximate the obstacle problem (2.1),(2.2) by

$$(2.28)_i \begin{cases} \underline{\text{Find } u^i_h \in K^i_h \text{ such that}} \\ \displaystyle\int_\Omega \nabla u^i_h \cdot \nabla(v_h-u^i_h)\,dx \geq \int_\Omega f(v_h-u^i_h)\,dx \quad \forall v_h \in K^i_h\ , \end{cases}$$

where i=1,2.

Proposition 2.2 : The approximate problems $(2.28)_i$ have a unique solution.

Concerning the convergence of the approximate solution we refer to GLOWINSKI [53] where $\lim_{h\to 0} \|u^i_h-u\|_{1,\Omega} = 0$ is proved, for i=1,2, assuming the usual angle condition. The error estimates analysis is much more complicated that with k=1, and we refer to BREZZI-HAGER-RAVIART [21] where under suitable assump-

tions on f, u, ψ, g and the free boundary, one proves that $\forall i=1,2$

$$\|u_h^i - u\|_{H^1(\Omega)} = O(h^{3/2-\varepsilon}) \ , \ \varepsilon>0 \text{ arbitrary small,}$$

if the angle condition holds.

2.1.8. Concluding Remarks. Further comments.

Non linear obstacle problems have been considered by WHITEMAN-NOOR [83]. Concerning the numerical solution of the approximate obstacle problems we refer to Sec. 3.1 where several methods of solutions are described.

2.2. The elasto-plastic torsion problem.

The problem we shall consider in this section is more complicated than the obstacle problem of Sec. 2.1. It is related to the elasto-plastic torsion of a cylindrical bar of infinite length. It is a fairly simple plasticity problem but it is of great interest from both theoretical and numerical points of view.

2.2.1. Formulation of the continuous problem.

Let Ω be a bounded domain of $\mathbf{R}^2$ with a smooth boundary $\Gamma = \partial\Omega$. We consider

$$(2.29) \quad \begin{cases} \underline{\text{Find}}\ u \in K\ \underline{\text{such that}} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq \int_\Omega f(v-u)\,dx \quad \forall\, v \in K, \end{cases}$$

where $f \in L^2(\Omega)$ and

$$(2.30) \qquad K = \{v \in H_0^1(\Omega) \ , \ |\nabla v| \leq 1 \text{ a.e.}\} \ .$$

We recall that

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} = \{v \mid v \in H^1(\Omega) \ , \ v=0 \text{ on } \partial\Omega\}.$$

Remark 2.2 : Since the bilinear form $a(\cdot,\cdot)$ occuring in (2.29) is symmetric ($a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx$), (2.29) is indeed equivalent to the minimization problem

$$(2.31)\quad \begin{cases} \underline{\text{Find}}\ u \in K\ \underline{\text{such that}} \\ J(u) \le J(v) \quad \forall v \in K, \end{cases}$$

where $J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv\, dx$.

2.2.2. Physical interpretation.

Let consider an infinitely long cylindrical bar of cross section Ω , where Ω is simply connected. Assume this bar is made of an isotropic, elastic, perfectly plastic material whose plasticity yield is given by the Von Mises criterion (see DUVAUT-LIONS [45, Ch. 5] , for a general discussion of plasticity phenomena). Starting from a zero-stress initial state, an increasing torsion moment is applied to the bar. The torsion is characterized by its torsion angle per unit length C. It follows then from the Haar-Karman principle that the stress field can be obtained through the solution of the following variational problem

$$(2.32)\qquad \underset{v\in K}{\text{Min}} \left\{\frac{1}{2}\int_\Omega |\nabla v|^2 dx - C\int_\Omega v\, dx\right\} ,$$

which is a particular case of (2.29), (2.31), with f=C.

The stress vector σ in Ω is obtained from u by $\sigma = \nabla u$. Hence u appears as a stress potential.

Remark 2.3 : If Ω is not simply connected, the formulation of the elasto-plastic problem has to be modified and we refer to GLOWINSKI-LANCHON [56], GLOWINSKI [53, Ch. 4, Sec. 3.2] for the new formulation.

2.2.3. Existence and Uniqueness results. Regularity and further properties.

The condition of Sec. 1 being fulfilled we can apply Theorem 1.1., then

Proposition 2.3 : Problems (2.29), (2.31) have a unique solution.

For the proof we refer to GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 3] . Concerning regularity properties it follows from BREZIS-STAMPACCHIA [19] that if $\partial\Omega$ is smooth (or Ω convex) and if $f \in L^p(\Omega)$ with $p \ge 2$ then the solution u of (2.29), (2.31) satisfies

$$u \in W^{2,p}(\Omega) \cap K.$$

If in particular, f = const. (as in (2.32)) then $u \in W^{2,p}(\Omega)$ for p arbitrary large. If for example Ω is a disk and f = const. then for f large enough $u \in W^{2,\infty}(\Omega) \cap H^s(\Omega)\ \forall\, s < \frac{5}{2}$, but $u \notin C^2(\overline{\Omega})$, $u \notin H^3(\Omega)$.

Remark 2.4 : If f = const., BREZIS-SIBONY [20] have proved that the solution u of (2.29), (2.31) is also the unique solution of the two-obstacles problem

$$(2.33) \qquad \underset{v \in K^*}{\text{Min}} \ \frac{1}{2}\int_\Omega |\nabla v|^2 dx - C\int_\Omega v\, dx\}$$

where

$$K^* = \{v \in H_0^1(\Omega),\ |v(x)| \leq \delta(x,\Gamma) \quad \text{a.e.}\}$$

with $\delta(x,\Gamma)$ = distance of x to $\Gamma = \partial\Omega$.

Remark 2.5 : For the free boundary aspect of the elasto-plastic problem we refer to e.g. GLOWINSKI-LIONS-TREMOLIERES [57]. Actually SHAW [80] has numerically solved (2.32) as a free boundary problem using finite difference approximations.

2.2.4. Finite element approximations of (2.29), (2.31).

We assume Ω polygonal, then we define $\mathcal{T}_h$ as in Sec. 2.1.6 and we introduce

$$V_{oh} = \{v_h \in C^0(\overline{\Omega}),\ v_h|_\Gamma = 0,\ v_h|_T \in P_1 \ \forall\, T \in \mathcal{T}_h\},$$

$$K_h = \{v_h \in V_h,\ |\nabla v_h| \leq 1 \quad \text{a.e.}\} = K \cap V_{oh}.$$

Then we approximate (2.29) by

$$(2.34) \qquad \begin{cases} \underline{\text{Find}}\ u_h \in K_h\ \underline{\text{such that}} \\ \int_\Omega \nabla u_h \cdot \nabla(v_h - u_h)\,dx \leq \int_\Omega f(v_h - u_h)\,dx \quad \forall\, v_h \in K_h. \end{cases}$$

It is clear that (2.34) has a unique solution.

Remark 2.6 : Since ∇v_h is piecewise constant, the condition $v_h \in K_h$ amounts to Card $(\mathcal{T}_h)$ quadratic constrains $(|\nabla v_h|^2 \leq 1$ on $T, \forall T \in \mathcal{T}_h)$. If instead of a piecewise linear approximation, one uses a piecewise quadratic, requiring $|\nabla v_h| \leq 1$ a.e. then $v_h \in K_h$ would amount to 3 Card$(\mathcal{T}_h)$ quadratic constraints (see GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 3] for more details).

Remark 2.7 : The numerical analysis of (2.29) via (2.33) is done, if f = const. in GLOWINSKI-LIONS-TREMOLIERES, loc. cit., Ch. 3.

Convergence analysis : Since $\overline{\mathcal{D}(\Omega) \cap K}^{H_o^1(\Omega)} = K$ (where $\mathcal{D}(\Omega) = \{v \in C^\infty(\overline{\Omega})$, v has a compact support in $\Omega\}$) we can prove using the general approximation results of Sec. 1 that

$$\lim_{h\to 0} \|u_h - u\|_{1,\Omega} = 0 \tag{2.35}$$

if the angle condition holds. For the proof of (2.35) and of the above density result see e.g. GLOWINSKI-LIONS-TREMOLIERES, loc. cit., Ch. 3 and GLOWINSKI [53, ch. 4, Sec. 3].

Moreover if $f \in L^p(\Omega)$ and $u \in W^{2,p}(\Omega)$ with $p > 2$, it is proved in FALK [46] (see also GLOWINSKI [53, Chap. 4, Sec. 3]) that

$$\|u_h - u\|_{1,\Omega} = O(h^{1/2-1/p}) . \tag{2.36}$$

In FALK [46] one also considers the case where Ω is non polygonal.

Remark 2.8 : If $\Omega \subset \mathbb{R}$ then $f \in L^2(\Omega)$ implies that $\|u_h - u\|_{1,\Omega} = O(h)$, instead of (2.36). This result is related to the fact that in the monodimensional case the piecewise linear interpolate of $v \in K$ is still in K, which is no longer true in $\mathbb{R}^2$.

The iterative solution of the approximate problems is discussed in Sec. 3.2.

2.3. Flow of a Bingham medium in a pipe.

We have considered in the two previous sections, examples of EVI's of the first kind. In the present section we shall discuss an EVI of the second kind related to the flow of a Bingham's viscous-plastic medium in a pipe. This section follows GLOWINSKI [54] and GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 5] (see also DUVAUT-LIONS [45, Ch. 6] for a more precise mechanical interpretation).

2.3.1. Formulation of the continuous problem.

Let Ω be a bounded domain of $\mathbf{R}^2$ with a smooth boundary $\Gamma = \partial\Omega$. Let define $j(\cdot)$ by

$$(2.37) \qquad j(v) = \int_\Omega |\nabla v| dx ;$$

$j(\cdot)$ is Lipschitz continuous but not differentiable.

Let consider now the following EVI of the second kind (with $f \in L^2(\Omega)$) :

$$(2.38) \quad \begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \mu\int_\Omega \nabla u \cdot \nabla(v-u) dx + gj(v) - gj(u) \geq \int_\Omega f(v-u) dx \quad \forall v \in H_0^1(\Omega) , \end{cases}$$

which is equivalent to

$$(2.39) \qquad \min_{v \in H_0^1(\Omega)} \{\frac{\mu}{2}\int_\Omega |\nabla v|^2 dx + gj(v) - \int_\Omega fv\, dx\} .$$

Assuming that $\mu > 0$ and $g \geq 0$ if follows from Theorem 1.2 of Sec. 1.4 that

Proposition 2.4 : The two equivalent problems (2.38),(2.39) have a unique solution.

2.3.2. Mechanical Interpretation.

If f = const. = C ($C > 0$ for example) it follows from LIONS-DUVAUT [45 , Ch. 6] that (2.38),(2.39) models the laminar stationary flow of a Bingham's viscous plastic fluid in a cylindrical pipe of cross section Ω, with $u(x)$ the velocity at x. The above constant C is the linear decay of pressure and μ, g are respectively the viscosity and the plasticity yield of the medium.

The above medium behaves like a viscous fluid (of viscosity μ) in

$$\Omega^+ = \{x | x \in \Omega \ , \ |\nabla u(x)| > 0\}$$

and like a rigid medium in

$$\Omega^0 = \{x | x \in \Omega, \ \nabla u(x) = 0\} .$$

We refer to MOSSOLOV-MIASNIKOV [73], [74], [75] for a detailed analysis of the properties of Ω^+ and Ω^o.

2.3.3. Regularity properties. Existence of multipliers.

Regularity properties : Concerning the regularity of the solution u of (2.38), (2.39), H. BREZIS [15] has proved that $u \in H^2(\Omega) \cap H_o^1(\Omega)$ and also if Ω is convex

$$(2.40) \qquad \|u\|_{H^2(\Omega)} \leq \frac{\gamma(\Omega)}{\mu} \|f\|_{L^2(\Omega)} .$$

If Ω is a disk and f = const. then we have $u \in W^{2,\infty}(\Omega) \cap H^s(\Omega) \ \forall s < \frac{5}{2}$, but if g is small enough $u \notin C^2(\bar{\Omega})$, $u \notin H^3(\Omega)$.

Let mention also that if g is large enough then u=0.

A characterization involving multipliers : Let define

$$\Lambda = \{q \mid q \in L^2(\Omega) \times L^2(\Omega) \ , \ |q(x)| \leq 1 \text{ a.e.}\}$$

where $|q| = \sqrt{q_1^2+q_2^2}$. It follows then from e.g. GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 5] that the solution u of (2.38),(2.39) is characterized by the existence of p such that

$$(2.41) \quad \begin{cases} \mu\int_\Omega \nabla u \cdot \nabla v \, dx + g\int_\Omega p \cdot \nabla v \, dx = \int_\Omega fv \, dx \quad \forall v \in H_o^1(\Omega) , \\ u \in H_o^1(\Omega) , \end{cases}$$

$$(2.42) \quad \begin{cases} p \cdot \nabla u = |\nabla u| \text{ a.e.}, \\ p \in \Lambda . \end{cases}$$

2.3.4. Finite Element approximations.

Since the regularity of the solution of (2.38), (2.39) is usually low we just concentrate on piecewise linear approximations. Let assume that Ω is a polygonal domain. Then we define $\mathcal{T}_h$ as in Sec. 2.1.6. and V_{oh} as in Sec. 2.2.4., and we approximate (2.38), (2.39) by

$$(2.43)\quad \begin{cases} \underline{\text{Find}}\ u_h \in V_{oh}\ \underline{\text{such that}}\ \forall v_h \in V_{oh} \\ \mu\int_\Omega \nabla u_h \cdot \nabla(v_h - u_h)\,dx + g\int_\Omega |\nabla v_h|\,dx - g\int_\Omega |\nabla u_h|\,dx \geq \int_\Omega f(v_h-u_h)\,dx\ . \end{cases}$$

The approximate problem (2.43) has clearly a unique solution. Concerning the convergence of u_h to u as $h \to 0$ we have

<u>Theorem 2.3</u> : <u>Assume that the angles of</u> $\mathcal{T}_h$ <u>are uniformly bounded from below by</u> $\theta_o > 0$, <u>as</u> $h \to 0$, <u>then</u>

$$(2.44)\quad \lim_{h\to 0} \|u_h - u\|_{1,\Omega} = 0.$$

<u>If furthermore</u> $u \in H^2(\Omega)$, <u>then</u>

$$(2.45)\quad \|u_h - u\|_{1,\Omega} = 0(h^{1/2}).$$

<u>Proof</u> : We follow GLOWINSKI [54]. Taking $v_h = 0$ in (2.43) we obtain

$$(2.46)\quad |u_h|^2_{1,\Omega} \leq \frac{1}{\mu} \|f\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)} \quad \forall h\ .$$

Since Ω is <u>bounded</u> $|v|_{1,\Omega} = (\int_\Omega |\nabla v|^2 dx)^{1/2}$ defines on $H^1_o(\Omega)$ a norm equivalent to $\|v\|_{1,\Omega}$. We have moreover

$$(2.47)\quad \begin{cases} \|v\|_{L^2(\Omega)} \leq \dfrac{1}{\lambda_o^{1/2}} |v|_{1,\Omega} \quad \forall v \in H^1_o(\Omega)\ , \\ \lambda_o = \text{smallest eigenvalue of } -\Delta \text{ over } H^1_o(\Omega). \end{cases}$$

It follows then from (2.46),(2.47) that

$$(2.48)\quad |u_h|_{1,\Omega} \leq \frac{1}{\mu\lambda_o^{1/2}} \|f\|_{L^2(\Omega)}\ .$$

In other respect we do have

$$\begin{cases} \mu\int_\Omega \nabla u_h \cdot \nabla(v_h - u_h)\,dx + g\, j(v_h) - g j(u_h) \geq \int_\Omega f(v_h - u_h)\,dx \quad \forall v_h \in V_{oh}\ , \\ \mu\int_\Omega \nabla u \cdot \nabla(u_h - u)\,dx + g j(u_h) - g j(u) \geq \int_\Omega f(u_h - u)\,dx\ , \end{cases}$$

and hence by addition we obtain

$$(2.49)\qquad \mu|u_h-u|^2_{1,\Omega} \le gj(v_h)-gj(u)+\mu\int_\Omega \nabla u_h\cdot\nabla(v_h-u)\,dx-\int_\Omega f(v_h-u)\,dx\ \ \forall v_h \in V_h .$$

From (2.48),(2.49) and

$$(2.50)\qquad j(v) = \int_\Omega |\nabla v|\,dx \le \sqrt{\text{meas.}(\Omega)}\ \ |v|_{1,\Omega}\qquad \forall v \in H^1_o(\Omega)$$

we obtain

$$(2.51)\qquad |u_h-u|^2_{1,\Omega} \le \frac{1}{\mu}\left(g\sqrt{\text{meas.}(\Omega)} + \frac{2}{\sqrt{\lambda_o}}\|f\|_{L^2(\Omega)}\right)|v_h-u|_{1,\Omega}\ \forall v_h \in V_h .$$

Let $\phi \in \mathcal{D}(\Omega)$; we denote by $\pi_h\phi$ the V_{oh}-interpolate of ϕ on $\mathcal{C}_h$, i.e.

$$\begin{cases} \pi_h\phi \in V_{oh} , \\ \pi_h\phi\ (P) = \phi(P)\ \ \forall\, P \text{ vertex of } \mathcal{C}_h . \end{cases}$$

Since the angle condition holds we have

$$(2.52)\qquad |\pi_h\phi-\phi|_{1,\Omega} \le C\|\phi\|_{H^2(\Omega)}\,h \qquad \forall \phi \in \mathcal{D}(\Omega)$$

with C independent of h and ϕ.

From (2.51),(2.52) and from the triangular inequality we obtain, taking $v_h = \pi_h\phi$ in (2.51)

$$(2.53)\qquad \begin{cases} |u_h-u|^2_{1,\Omega} \le \frac{1}{\mu}\left(g\sqrt{\text{meas.}(\Omega)} + \frac{2}{\sqrt{\lambda_o}}\|f\|_{L^2(\Omega)}\right)\left(|\phi-u|_{1,\Omega}+C\|\phi\|_{H^2(\Omega)}h\right) \\ \forall \phi \in \mathcal{D}(\Omega) . \end{cases}$$

Since $\mathcal{D}(\Omega)$ is <u>dense</u> in $H^1_o(\Omega)$, (2.53) implies clearly (2.44). To prove (2.45) we use directly (2.51) ; indeed if $u \in H^2(\Omega) \cap H^1_o(\Omega)$, then $u \in C^o(\overline{\Omega})$ and $\pi_h u$ can be defined. We have furthermore

$$(2.54)\qquad |\pi_h u-u|_{1,\Omega} \le C\ h\ \|u\|_{H^2(\Omega)} .$$

Then taking $v_h = \pi_h u$ in (2.51) it is clear that (2.54) implies (2.45). Q.E.D.

Remark 2.9 : Assuming some extra-regularity assumptions (satisfied if Ω is a disk and if f = const.), it is shown in GLOWINSKI [53, Ch. 5], [54] that

$$\|u_h - u\|_{1,\Omega} = 0(|\log h|^{1/2} h) .$$

Iterative methods for solving (2.38) and (2.43) are described in Sec. 3.3.

2.4. Error estimates of optimal order for the elasto-plastic torsion and Bingham's flow problems via a new formulation.

In this Section we shall briefly describe some of the results of FALK-MERCIER [49] , who via a new variational formulation of the problems in Sec. 2.2, 2.3 have obtained error estimates of optimal order.

2.4.1. The elasto-plastic torsion problem.

We consider again the elasto-plastic problem of Sec. 2.2. We assume that Ω is simply connected (which corresponds precisely to the physical problem). Then

Proposition 2.5. : Assume that Ω is simply connected, then the variational problem (2.29) is equivalent to

$$(2.55) \quad \begin{cases} \text{Find } p \in \Lambda \cap H \text{ such that} \\ \int_\Omega p\cdot(q-p)\,dx \geq \int_\Omega \phi\cdot(q-p)\,dx \quad \forall q \in \Lambda \cap H \end{cases}$$

where $\phi = \{\phi_1, \phi_2\}$ is any solution of $f = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}$ and where

$$(2.56) \quad \Lambda = \{q \in L^2(\Omega) \times L^2(\Omega),\ |q| \leq 1 \text{ a.e. on } \Omega\}$$

$$(2.57) \quad H = \{q \in L^2(\Omega) \times L^2(\Omega)\ ,\ \int_\Omega q\cdot\nabla w\,dx = 0 \ \ \forall w \in H^1(\Omega)\} .$$

The solutions u of (2.29) and p of (2.55) are related by

$$(2.58) \quad p = \{\frac{\partial u}{\partial x_2}\ ,\ -\frac{\partial u}{\partial x_1}\} .$$

Remark 2.10 : It is clear that $q \in H$ is equivalent to

$$\begin{cases} q \in L^2(\Omega) \times L^2(\Omega), \\ \nabla \cdot q = 0 \text{ a.e. on } \Omega, \ q \cdot n = 0 \text{ a.e. on } \partial\Omega \end{cases}$$

where n is the outward unit normal vector at Ω.

Remark 2.11 : If f = const. = C, one can take

$$\phi_1 = Cx_2 \ , \ \phi_2 = 0 \ .$$

If f is not constant and $\partial\Omega$ smooth or Ω convex it is always possible to construct from $f \in L^2(\Omega)$, $\phi_1, \phi_2 \in H^1(\Omega)$ such that $f = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}$. For instance we solve

$$\begin{cases} \Delta u_o = f \text{ on } \Omega, \\ u_o \in H_o^1(\Omega) \end{cases}$$

which produces $u_o \in H^2(\Omega) \cap H_o^1$. It suffices then to take

$$\phi_1 = \frac{\partial u_o}{\partial x_2} \ , \phi_2 = - \frac{\partial u_o}{\partial x_1} \ .$$

The approximate problem : For simplicity we assume that Ω is a polygonal bounded convex domain of $\mathbb{R}^2$; let $\mathcal{T}_h$ be a triangulation of Ω like in the above sections. We approximate $H^1(\Omega)$, $L^2(\Omega) \times L^2(\Omega)$ and H by respectively

$$V_h = \{v_h \in C^o(\bar{\Omega}) \ , \ v_h|_T \in P_1 \ \ \forall T \in \mathcal{T}_h\},$$

$$L_h = \{q_h \in L^2(\Omega) \times L^2(\Omega) \ , \ q_h|_T \in \mathbb{R}^2 \ \ \forall T \in \mathcal{T}_h\} \ ,$$

$$H_h = \{q_h \in L_h, \int_\Omega q_h \cdot \nabla w_h \, dx = 0 \ \ \forall w_h \in V_h\} \ .$$

Then we approximate (2.55) by

$$(2.59) \quad \begin{cases} \underline{\text{Find}} \ p_h \in \Lambda \cap H_h \ \underline{\text{such that}} \\ \int_\Omega p_h \cdot (q_h - p_h) dx \geq \int_\Omega \phi \cdot (q_h - p_h) dx \ \ \forall \ q_h \in \Lambda \cap H_h . \end{cases}$$

The approximate problem (2.59) has a unique solution and it is proved in FALK-MERCIER [49] that if some convenient properties of u and p hold together with the usual angle condition, then

(2.60) $$\|p_h - p\|_{L^2(\Omega)\times L^2(\Omega)} = O(h) .$$

If for example f = const., which correspond to the physical problem, then (2.60) holds. We refer to FALK-MERCIER, loc. cit., for more details.

Concluding remark : In conclusion by a change of formulation an error estimate of optimal order have been obtained. It seems however that (2.59) is more complicated to solve numerically than (2.34).

2.4.2. The Bingham's flow problem.

We consider now the Bingham's flow problem of Sec. 2.3. Then we have

Proposition 2.6 : The variational problem (2.38) is equivalent to

(2.61) $$\begin{cases} \text{Find } p \in H \text{ such that} \\ \mu\int_\Omega p\cdot(q-p)dx + gj(q) - gj(q) \geq \int_\Omega \phi\cdot(q-p)dx \quad \forall q \in H \end{cases}$$

where ϕ and H are like in the statement of Prop. 2.5 and where

$$j(q) = \int_\Omega |q|dx .$$

The solutions u of (2.38) and p of (2.61) are related by

$$p = \{\frac{\partial u}{\partial x_2} , - \frac{\partial u}{\partial x_1} \} .$$

The approximate problem : With V_h, L_h, H_h as in Sec. 2.4.1. we approximate (2.61) by

(2.62) $$\begin{cases} \text{Find } p_h \in H_h \text{ such that} \\ \mu\int_\Omega p_h\cdot(q_h-p_h)dx + gj(q_h) - gj(p_h) \geq \int_\Omega \phi\cdot(q_h-p_h)dx \quad \forall q_h \in H_h. \end{cases}$$

The approximate problem (2.62) has a unique solution and under suitable assumptions on u and p it follows from FALK-MERCIER [49] that

$$\|p_h - p\|_{L^2(\Omega)\times L^2(\Omega)} = O(h)$$

(we still assume that the angle condition holds).
The concluding remark of Sec. 2.4.1. still holds for (2.62).

2.5. Further problems.

We have not considered in this paper the numerical analysis via finite elements of problems like

$$(2.63)\quad \begin{cases} \underline{\text{Find}}\ u \in K\ \underline{\text{such that}} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla (v-u)\,dx \geq \int_\Omega f(v-u)\,dx \quad \forall v \in K \end{cases}$$

with

$$K = \{v \in H^1(\Omega)\ ,\ v|_{\Gamma_o} = g\ ,\ v \geq \psi \quad \text{a.e. on } \Gamma_1\}$$

where Γ_o and Γ_1 are such that $\Gamma_o \cap \Gamma_1 = \emptyset$, $\Gamma_o \cup \Gamma_1 = \partial\Omega$, or like

$$(2.64)\quad \begin{cases} \underline{\text{Find}}\ u \in H^1(\Omega)\ \underline{\text{such that}} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla (v-u)\,dx + j(v) - j(u) \geq \int_\Omega f(v-u)\,dx \quad \forall\, v \in H^1(\Omega)\ , \end{cases}$$

with

$$j(v) = g\int_{\partial\Omega} |v|\,d\Gamma\ .$$

For finite element approximations of these problems by conforming methods we refer to GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 4] , GLOWINSKI [53, Ch. 4], SCARPINI-VIVALDI [79], MOSCO [70], BREZZI-HAGER-RAVIART [21], etc...

3. - ITERATIVE METHODS FOR SOLVING THE APPROXIMATE VARIATIONAL INEQUALITIES OF SEC. 2.

In this section we shall give some brief indications on the actual solution of the approximate problems of Sec. 2 by iterative methods. For more details and other applications, we refer our readers to GLOWINSKI-LIONS-TREMOLIERES [57] , TREMOLIERES [81], GLOWINSKI [55], MERCIER [69], the references therein and the references below. In fact the methods to follows are closely related to Non Linear Programming.

3.1. Iterative solution of the obstacle problem.

3.1.1. Orientation

The obstacle problem of Sec. 3.1 and its variants (possibly involving non symmetric $a(\cdot,\cdot)$) may be solved by several methods. We shall concentrate on S.O.R. with truncation (see Sec. 3.1.2), duality (see Sec. 3.1.3) and give some indication on the use of penalty methods in Sec. 3.1.4.

Actually solution of discrete variational inequalities based on the so-called Complementarity Methods have been studied these last years ; in that direction we refer to e.g. COTTLE [38], [39], COTTLE-GOLUB-SACHER [40], MOSCO-SCARPINI [71] . In our opinion these methods are less effective (at least for most discrete EVI's) than the methods of Sec. 3.1.2, 3.1.3 and will not be considered here.

3.1.2. Methods of S.O.R. with truncation.

These methods have been by far the most popular for solving discrete obstacle problems and are in our opinion the simplest to program and the most economical in term of computer storage.

The various discrete obstacle problems we have discussed in Sec. 2.2 are in fact particular cases of

$$(3.1) \qquad \underset{\underset{\sim}{y} \in C}{\mathrm{Min}} \left\{ \frac{1}{2} (\underset{\sim}{A}\underset{\sim}{y},\underset{\sim}{y})-(\underset{\sim}{b},\underset{\sim}{y}) \right\}$$

where in (3.1) $,\underset{\sim}{y} = \{y_1,\dots y_N\} \in \mathbf{R}^N$, $\underset{\sim}{A}$ is a $N \times N$ positive definite symmetric matrix, $(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{i=1}^{N} x_i y_i$, $\underset{\sim}{b} \in \mathbf{R}^N$ and

(3.2) $\quad C = \{\underset{\sim}{y} \in \mathbf{R}^N \;,\; a_i \le y_i \le b_i \quad \forall i=1,\ldots N\}$

with $a_i \le b_i$ $\forall i=1,\ldots N$. Some of the a_i's (resp. b_i's) can possibly be equal to $-\infty$ (resp. $+\infty$).

If $\underset{\sim}{A} = (a_{ij})_{1 \le i,j \le N}$ then a typical S.O.R. + truncation algorithm is

(3.3) $\quad \underset{\sim}{x}^o \in \mathbf{R}^N$, arbitrary given,

assuming $\underset{\sim}{x}^n$ known, one computes $\underset{\sim}{x}^{n+1}$ from $\underset{\sim}{x}^n$, component by component as follows

For $i=1,\ldots N$, compute

$$(3.4)_i \qquad x_i^{n+1/2} = \frac{1}{a_{ii}} \{b_i - \sum_{j<i} a_{ij} x_j^{n+1} - \sum_{j>i} a_{ij}\, x_j^n\}$$

$$(3.5)_i \qquad x_i^{n+1} = P_i(x_i^n + \omega(x_i^{n+1/2} - x_i^n))$$

with $P_i(y_i) = \sup\,(a_i,\ \inf(b_i, y_i))$.

About the convergence we have

Theorem 3.1 : If $0 < \omega < 2$, then $\forall \underset{\sim}{x}^o \in \mathbf{R}^N$,

$$\lim_{n \to +\infty} \underset{\sim}{x}^n = \underset{\sim}{x}$$

where $\underset{\sim}{x}$ is the unique solution of (3.1).

For the proof of Theorem 3.1 we refer to CRYER [42], CEA-GLOWINSKI [29], COMINCIOLI [36], GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 2]. In some of the above references generalization to block algorithms in Hilbert spaces are also considered.

Remarks on the choice of ω : It appears that the optimal value ω_{opt} of ω (i.e. this giving the fastest convergence for a given norm) is a function of C and b. Therefore for the discrete obstacle problems discussed above it will be a function of f, g, ψ.

In practice several strategies can be used ; one can either use the optimal ω of the corresponding linear problem ($\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$), or apply the Young's method (see VARGA [83], YOUNG [84]). Actually the last method has given very good results even for matrices $\underset{\sim}{A}$ which are not M-matrices.

Remark 3.1 : In practice, when using S.O.R. with truncation to solve discrete obstacle problems, the first components of $\underset{\sim}{x}^n$ to converge are those for which the discrete solution coincides with ψ . It appears in fact that most of the computational time is used to compute the approximate solution of $-\Delta u = f$ on $\Omega^+ = \{x \in \Omega,\ u(x) > \psi(x)\}$, with $u|_{\partial\Omega^+} = \psi|_{\partial\Omega^+}$ as boundary conditions. And indeed we have observed that the optimal value of ω corresponds to the optimal choice for the approximate solution of the corresponding linear Dirichlet problem on Ω^+.

Concluding comments : The main advantages of S.O.R. methods with truncation are that :

- they are easy to program
- they require few computer storage,
- they have however some drawbacks which are that they are mainly limited to second order potential problems. Indeed they usually show a fairly slow convergence when applied to the solution of obstacle problems related to Δ^2 or to elasticity operators (the block variants of algorithm (3.3-(3.5) are not so easy to program since the truncation has to be replaced by a more complicated projection step).

3.1.3. Solution of the obstacle problem by duality methods.

Several dual problems may be associated to the obstacle problem (2.1),(2.2) (see Sec. 4 below for one of them). Among these dual formulations the following is well suited for computations.

Let define

$$\Lambda = \{\mu \in L^2(\Omega),\ \mu \geq 0 \text{ a.e.}\} ,$$

$$V_g = \{v \in H^1(\Omega) \ ,\ v=g \text{ on } \partial\Omega\}$$

and a lagrangian functional $\mathscr{L} : H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by

$$\mathscr{L}(v,\mu) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv\ dx - \int_\Omega \mu(v-\psi)dx .$$

Assume that the solution u of (2.1),(2.2) belongs to $H^2(\Omega)$, then $\lambda = -\Delta u - f$ is the unique solution of the dual problem

(3.6) $$\underset{\mu\in\Lambda}{\text{Max}}\ \underset{v\in V_g}{\text{Min}}\ \mathscr{L}(v,\mu) ;$$

moreover we can easily prove that $\{u,\lambda\}$ is the unique saddle-point of $\mathscr{L}$ over $V_g \times \Lambda$.

From these properties we can use the Uzawa's algorithm discussed in GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 2] which takes the following form :

(3.7) $\lambda^0 \in L^2(\Omega)$, arbitrary given ($\lambda^0 = 0$ for ex.)

then for $n \geq 0$, assuming λ^n known we obtain u^n and λ^{n+1} by

(3.8) $$\begin{cases} -\Delta u^{n+1} = f + \lambda^n \text{ in } \Omega , \\ u^{n+1} = g \text{ on } \partial\Omega, \end{cases}$$

(3.9) $$\lambda^{n+1} = P_+(\lambda^n + \rho(\psi - u^n)) , \rho > 0 ,$$

where

$$P_+(\mu) = \text{Sup}(0,\mu)$$

Using GLOWINSKI-LIONS-TREMOLIERES [57 , Ch. 2] we can prove

Theorem 3.2 : Assume that $u \in H^2(\Omega)$, then $\forall \lambda^0 \in L^2(\Omega)$ we have

$$\lim_{n\to+\infty} ||u^n - u||_{H^1(\Omega)} = 0$$

if $0 < \rho < 2\beta_0$

where β_o is the smallest eigenvalue in

$$\begin{cases} -\Delta w = \beta w \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases}$$

Comments :

1) If one uses $\lambda^o = 0$ we have observed that the smallest is $\Omega^o = \{x \in \Omega,\ u(x)=\psi(x)\}$, the fastest is the convergence.

2) The above method (in its discrete form) is very well suited to users having at their disposal finite difference or finite element elliptic solvers.

3) Another advantage of this duality approach is that it gives directly λ whose Mechanical interpretation is interesting since it is the reaction force of the obstacle on the membrane.

4) Variants of the above algorithm have been successfully used with Δ replaced by Δ^2 or elasticity operators.

3.1.4. Solution of the obstacle problem by penalty methods.

Like in Sec. 3.1.3. we shall focus our attention to the continuous obstacle problem whose formalism is simpler.

With V_g like in Sec. 3.1.3. we consider

$$(3.10) \qquad \underset{v \in V_g}{\mathrm{Min}} \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega fv\, dx + j_\varepsilon(x) \right\}$$

with

$$(3.11) \qquad j_\varepsilon(v) = \frac{1}{2\varepsilon} \int_\Omega |(\psi - v)^+|^2 dx$$

where $q^+ = \sup(0,q)$.

The minimization problem (3.10) is in fact equivalent to the Non Linear Dirichlet problem

$$(3.12)\quad \begin{cases} -\Delta u_\varepsilon - \frac{1}{\varepsilon}(\psi-u_\varepsilon)^+ = f \text{ in } \Omega, \\ u_\varepsilon = g \text{ on } \partial\Omega. \end{cases}$$

Concerning the convergence of u_ε to the solution u of (2.1) (2.2) we can prove that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{H^1(\Omega)} = 0 .$$

The non linear problem (3.10), (3.12) (in fact its discrete variants) can be solved by various methods, like e.g. Non Linear S.O.R., Conjugate Gradient with Scaling (cf. CONCUS-GOLUB-O'LEARY [37], DOUGLAS-DUPONT [44]) using as scaling operator a discrete form of $-\Delta$, or an operator obtained by Incomplete Cholesky Decomposition (see MEIRINJK-VAN DE VORST [68] for details).

Comments : The main inconvenient of that penalty method is that it requires a small ε to have $u_\varepsilon - u$ small. Then (3.10),(3.12) are ill-conditioned problems. However this technique has been successfully used in Optimal Control or Optimal Design problems in which the state equation is replaced by a variational inequality.

3.1.5. Other methods.

The above obstacle problem can also be solved by Conjugate gradient with truncation methods (see e.g. TREMOLIERES [81]) or by using a discrete time dependent approach requiring at each time step the solution of a problem of similar type but better conditioned.

We can also use augmented lagrangian methods (in this direction we refer to CHAN-GLOWINSKI [32]).

3.2. Iterative solution of the elasto-plastic torsion problem.

In this section we just describe one algorithm and send the reader to GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 3], CEA-GLOWINSKI-NEDELEC [31], FORTIN-GLOWINSKI [51, Ch. 3] and GLOWINSKI-MARROCCO [58] for more details and/or other methods.

The problem under consideration is (2.29) ; we associate to this problem a lagrangian functional $\mathcal{L} : H_0^1(\Omega) \times L^\infty(\Omega) \to \mathbb{R}$ defined by

(3.13) $$\mathcal{L}(v,\mu) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv\, dx + \int_\Omega \mu(|\nabla v|^2 - 1)dx \ .$$

If $\Lambda = \{\mu \in L^\infty(\Omega),\ \mu \geq 0 \text{ a.e. }\}$, then we can easily prove that if $\{u,\lambda\}$ is a saddle-point of $\mathcal{L}$ over $H_0^1(\Omega)\times\Lambda$, then u is the solution of (2.29). If f = const. the existence of such a saddle-point has been proved by H. BREZIS [16], for more general f the situation is not clear at the moment. However the existence of such saddle-point is proved for the finite difference or finite element approximation of the elasto-plastic problem (see GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 3] and CEA-GLOWINSKI-NEDELEC [31] for further information). From the above properties it is then natural to use the following algorithm to compute the saddle-points of $\mathcal{L}$ and therefore u :

(3.14) $\lambda^0 \in \Lambda$, arbitrarily given ,

then for $n \geq 0$ compute u^n and λ^{n+1} by

(3.15) $$\begin{cases} -\nabla\cdot(1+\lambda^n)\nabla u^n = f \text{ in } \Omega\ , \\ u^n|_{\partial\Omega} = 0, \end{cases}$$

(3.16) $$\lambda^{n+1} = P_+(\lambda^n + \rho(|\nabla u^n|^2 - 1))\ ,\ \rho > 0\ ,$$

with P_+ like in Sec. 3.1.3.

For the convergence of (3.14)-(3.16) see the two references above.

We observe that the solution of (3.15) require at each iteration the solution of a Dirichlet problem whose right-hand side depends upon n (via λ^n). However (3.14)-(3.16) appears as an efficient algorithm for solving (2.29).

3.3. Iterative solution of the Bingham flow problem in a pipe.

We consider in this section the iterative solution of (2.38),(2.39). Actually the simplest method to solve this problem is based on the characterization (2.41),(2.42). The algorithm is the following

(3.17) $p^0 \in L^2(\Omega) \times L^2(\Omega)$ is arbitrarily given ($p^0 = 0$ for ex.)

then for $n \geq 0$ one defines u^n and λ^{n+1} from λ^n by

$$
(3.18) \quad \begin{cases} -\mu\Delta u^n = f + g\, \nabla\cdot p^n \ \underline{\text{in}}\ \Omega, \\ u^n|_{\partial\Omega} = 0\ , \end{cases}
$$

$$
(3.19) \quad p^{n+1} = P_\Lambda\,(p^n + \rho\nabla u^n)\ ,\ \rho > 0\ ,
$$

where $\Lambda = \{q \in L^2(\Omega) \times L^2(\Omega)\ ,\ |q(x)| \le 1 \text{ a.e. }\}$ and

$$
P_\Lambda(q) = \frac{q}{\sup(1,|q|)} \quad \forall\, q \in L^2(\Omega) \times L^2(\Omega)\ .
$$

It follows then from CEA-GLOWINSKI [30] and GLOWINSKI-LIONS-TREMOLIERES [57, Ch. 2 and 5]

Theorem 3.3 : *Assume that we have*

$$
0 < \rho < \frac{2}{g}
$$

then $\forall\, p^o \in L^2(\Omega) \times L^2(\Omega)$ *we have*

$$
\lim_{n\to+\infty} \|u^n - u\|_{H^1(\Omega)} = 0\ ,
$$

$$
\lim_{n\to+\infty} p^n = p \ \underline{\text{in}}\ L^\infty(\Omega) \times L^\infty(\Omega)\ \underline{\text{weak-}\star}\ ,
$$

where u *is the solution of* (2.38),(2.39) *and where* p is such that {u,p} obeys (2.41),(2.42).

Remark 3.2 : The above function p is actually a solution of the dual problem

$$
\underset{q\in\Lambda}{\text{Max}}\ \underset{v\in H^1_o(\Omega)}{\text{Min}}\ \{\frac{\mu}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv\ dx + g\int_\Omega q\cdot\nabla v\ dx\}\ .
$$

Remark 3.3 : More efficient algorithms based on augmented lagrangian techniques are described in GABAY-MERCIER [52], FORTIN [50], FORTIN-GLOWINSKI [51], GLOWINSKI-MARROCCO [58]. These algorithms are more complicated to handle than (3.17)-(3.19) which is definitely the simpler efficient method to solve (2.38), (2.39) and their discrete variants.

4. - APPROXIMATION OF THE OBSTACLE PROBLEM BY MIXED FINITE ELEMENT METHODS.

4.1. Orientation.

Following BREZZI-HAGER-RAVIART [22] we shall consider in this section mixed finite element approximations of the obstacle problem (2.1),(2.2) (see also for related studies HLAVACHEK [61]).

For more complicated variational inequalities, solved also by mixed finite element methods we refer to, e.g., JOHNSON [62], [63], BREZZI-JOHNSON-MERCIER [23], BEGIS-GLOWINSKI [8].

4.2. A dual formulation of the obstacle problem.

It is clear that (2.1),(2.2) has also the following formulation

$$\min_{\{v,q\}\in\mathcal{K}} \left\{ \frac{1}{2}\int_\Omega |q|^2 dx - \int_\Omega fv\, dx \right\} \tag{4.1}$$

where

$$\mathcal{K} = \{\{v,q\}\in H^1(\Omega)\times(L^2(\Omega))^N \ ,\ q=\nabla v\ ,\ v=g \text{ on } \partial\Omega,\ v\geq\psi \ \text{ a.e. on } \Omega\}. \tag{4.2}$$

We associate to (4.1),(4.2) the lagrangian

$$\mathcal{L}(v,q,\mu) = \frac{1}{2}\int_\Omega |q|^2 dx - \int_\Omega fv\, dx + \int_\Omega \mu\cdot(\nabla v - q)\, dx$$

and then consider the dual problem of (2.1),(2.2) related to $\mathcal{L}$, i.e.

$$\max_{\mu\in(L^2(\Omega))^N}\ \min_{\{v,q\}\in\widetilde{\mathcal{K}}} \mathcal{L}(v,q,\mu)\ . \tag{4.3}$$

where

$$\widetilde{\mathcal{K}} = \{\{v,q\}\in H^1(\Omega)\times L^2(\Omega)\ ,\ v=g \text{ on } \partial\Omega,\ v\geq\psi \ \text{ a.e. on } \Omega\}\ .$$

Fortunately the explicit form of (4.3) is known and is as follows

$$\min_{q\in C} \left\{ \frac{1}{2}\int_\Omega |q|^2 dx + \int_\Omega \psi\, \nabla\cdot q\, dx - \int_\Gamma g\, q\cdot n\, d\Gamma \right\} \tag{4.4}$$

with

(4.5) $C = \{q \in H(\text{div},\Omega),\ \nabla\cdot q + f \le 0 \text{ a.e. on } \Omega\}$

where

$$H(\text{div},\Omega) = \{q \in (L^2(\Omega))^N \ ,\ \nabla\cdot q \in L^2(\Omega)\} \ .$$

Conversely (4.4) has a unique solution p such that $p = \nabla u$ where u is the solution of the obstacle problem (2.1), (2.2).
Define

$$\Lambda = \{\mu \in L^2(\Omega) \ ,\ \mu \ge 0 \text{ a.e. on } \Omega\}$$

then

(4.6) $$q \in C \Leftrightarrow q \in H(\text{div},\Omega) \ ,\ \int_\Omega (\nabla\cdot q + f)\mu \, dx \le 0 \quad \forall \mu \in \Lambda \ .$$

4.3. <u>A mixed approximation of the dual problem</u> (4.4),(4.5)

We still assume that Ω is a polygonal domain of $\mathbf{R}^2$ and $\mathcal{T}_h$ a standard triangulation of Ω . Let consider the following approximations of $H(\text{div},\Omega)$, $L^2(\Omega)$, Λ, C respectively

$H_h = \{q_h \in L^2(\Omega) \times L^2(\Omega) \ ,\ q_h\cdot n|_{\partial T} \in P_k$ sidewise where n is the unit normal along ∂T, $\nabla\cdot q_h|_T \in P_k\}$

(from the above properties $H_h \subset H(\text{div},\Omega)$) ,

$$L_h = \{\mu_h \in L^2(\Omega) \ ,\ \mu_h|_T \in P_k \quad \forall T \in \mathcal{T}_h\} \ ,$$

$$\Lambda_h = \Lambda \cap L_h = \{\mu_h \in L_h \ ,\ \mu_h \ge 0 \text{ a.e. on } \Omega\} \ ,$$

$$C_h = \{q_h \in H_h \ ,\ \int_\Omega (\nabla\cdot q_h + f)\mu_h \, dx \le 0 \quad \forall \mu_h \in \Lambda_h\} \ .$$

An approximate problem of (4.4), (4.5) is

$$(4.7)\quad \begin{cases} \underline{\text{Find}}\ p_h \in C_h\ \underline{\text{such that}} \\ \int_\Omega p_h\cdot(q_h-p_h)\,dx + \int_\Omega \psi\,\nabla\cdot(q_h-p_h)\,dx \geq \int_\Gamma g(q_h-p_h)\cdot n\,d\Gamma \quad \forall q_h \in C_h. \end{cases}$$

The problem (4.7) has clearly a unique solution.

An equivalent mixed formulation of (4.7) is

$$(4.8)\quad \begin{cases} \underline{\text{Find}}\ \{p_h,\lambda_h\} \in H_h \times \Lambda_h\ \underline{\text{such that}} \\ \int_\Omega p_h\cdot q_h\,dx + \int_\Omega (\lambda_h+\psi)\nabla\cdot q_h\,dx = \int_\Gamma g\, q_h\cdot n\,d\Gamma \quad \forall q_h \in H_h, \\ \int_\Omega (\nabla\cdot p_h+f)(\mu_h-\lambda_h)\,dx \leq 0\ \forall \mu_h \in \Lambda_h\ . \end{cases}$$

It follows from BREZZI-HAGER-RAVIART [22] that (4.8) has a unique solution with p_h solution of (4.7).

Concerning the convergence of p_h, λ_h as $h \to 0$, BREZZI-HAGER-RAVIART [22] have proved that if $(\mathcal{T}_h)_h$ is a regular family of triangulations, if $f \in L^2(\Omega)$, $u \in H^2(\Omega)$ (then $g \in H^{3/2}(\Gamma)$), and if $k=0$ then

$$\|p_h-\nabla u\|_{L^2(\Omega)} = \|\lambda_h-(u-\psi)\|_{L^2(\Omega)} = O(h).$$

The above authors have also proved that if $k=1$ and if $f \in L^\infty(\Omega)$, $u, \psi \in W^{2,\infty}(\Omega)$, $u \in W^{s,p}(\Omega)$ $\forall p \in [1,+\infty[$ and $s < 2 + \frac{1}{p}$, then

$$\|p_h-\nabla u\|_{L^2(\Omega)} = \|\lambda_h-(u-\psi)\|_{L^2(\Omega)} = O(h^{3/2-\varepsilon})$$

with $\varepsilon > 0$ arbitrary small.

Remark 4.1 : The numerical solution of (4.7),(4.8) is more complicated than this of the approximate problems of Sec. 2.1 corresponding to standard conforming (displacement) finite element approximations.

5. - FURTHER COMMENTS. CONCLUSION

The variational inequality methodology appears as an efficient tool for solving problems which may be originally formulated in a more classical way. Let us consider two examples of such a situation.

Example 1 - A family of mildly non linear elliptic problems.

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary, $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ $(= (H_0^1(\Omega))')$ a strongly elliptic isomorphism, $f \in H^{-1}(\Omega)$, $\phi : \mathbb{R} \to \mathbb{R}$, $\phi \in C^0$, ϕ non decreasing (we can always suppose that $\phi(o) = 0$). We consider then the following non linear elliptic problem (of monotone type)

$$Au + \phi(u) = f \; . \tag{5.1}$$

Let define $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, bilinear continuous and $H_0^1(\Omega)$-elliptic by

$$a(v,w) = \langle Av,w\rangle$$

where $\langle \cdot,\cdot \rangle$ denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Let define also $j : H_0^1(\Omega) \to \mathbb{R}$ by

$$j(v) = \int_\Omega \Phi(v)\,dx$$

where $\Phi(t) = \int_0^t \phi(\tau)d\tau$. It is clear that $j(\cdot)$ is convex, proper, l.s.c. Actually in view of solving (5.1) it is convenient to consider the following $(E.V.I.)_2$

$$(5.2) \quad \begin{cases} \underline{\text{Find}}\ u \in H_0^1(\Omega)\ \underline{\text{such that}} \\ a(u,v-u)+j(v)-j(u) \geq \langle f,v-u\rangle \quad \forall\, v \in H_0^1(\Omega). \end{cases}$$

In CHAN-GLOWINSKI [32] (see also GLOWINSKI [55]) one proves that (5.2) has a unique solution which is also the unique solution of (5.1) in $H_0^1(\Omega)$. We consider also in the two above references the approximation of (5.1),(5.2) by piecewise linear finite elements. This is precisely a situation in which the convergence results of Sec. 1.6.1. cannot be applied (directly at least), since $j(\cdot)$ is not continuous on $H_0^1(\Omega)$ in general. However the strong convergence of the approximate solution can be proved (see the above references for more details).

Example 2 - Transonic, potential flows.

This problem is far more important and difficult than the problem of Example 1. In fact it is not an elliptic problem unless the flow remains purely subsonic. The problem is to find a velocity potential ϕ defined on Ω (the domain of the flow) such that

$$(5.3)\quad \begin{cases} \nabla\cdot(\rho(\phi)\nabla\phi) = 0 \text{ in } \Omega, \\ + \text{ convenient } \underline{\text{boundary conditions}}, \end{cases}$$

with

$$\rho(\phi) = \rho_o\left(1 - \frac{|\nabla\phi|^2}{\frac{\gamma+1}{\gamma-1}C_\star^2}\right)^{1/\gamma-1},$$

where ρ_o = const., $\gamma = 1.4$ in air, $C_\star$ = critical velocity. The flow velocity is given by $\vec{v} = \nabla\phi$.

If Ω is not simply connected, ϕ has to obey a circulation condition given by the Kutta-Joukowsky condition (see LANDAU-LIFCHITZ [64] for more details, and also the references below). Actually the above relations are not sufficient to obtain only physical solutions, i.e. solutions without expansion shocks ; to avoid these non physical solutions an Entropy Condition has to be prescribed. We have found convenient to require

$$(5.4)\qquad (\Delta\phi)^+ \in L^p(\Omega) \quad \forall p > 1 .$$

The more common values of p are p=2 and p= $+\infty$. In order to solve (5.3), taking (5.4) into account we have introduced the variational problem

$$(5.5)\qquad \operatorname*{Min}_{w\in X}\left\{\int_\Omega |\nabla\phi(w)|^2 dx + \mu\int_\Omega |(\Delta w - C)^+|^2 dx\right\}$$

where X is a set (usually convex) of admissible velocity potentials, μ a positive parameter, C a given constant (or function) and $\phi(w)$ the solution of the elliptic problem

$$
(5.6) \quad \begin{cases} \Delta\phi(w) = \nabla\cdot(\rho(w)\nabla w) \text{ in } \Omega, \\ \\ + \text{ boundary conditions } + \text{ (possibly) Kutta-Joukowsky conditions.} \end{cases}
$$

In fact (5.5),(5.6) is a least square formulation of (5.3) taking (5.4) into account. The problem (5.5) which is indeed a non linear fourth order variational problem has been solved by a mixed finite element method coupled to a conjugate gradient algorithm with scaling. For further details we refer to GLOWINSKI-PIRONNEAU [59], [60], BRISTEAU [24], [25], BRISTEAU-GLOWINSKI-PERRIER-PERIAUX-PIRONNEAU-POIRIER [26] where references to other methods for solving (5.3) are also given.

To conclude this survey on Elliptic Variational Inequalities, let us mention several books or reports relevant to the subject :

- DUVAUT-LIONS [45] for the mathematical aspect and applications to Mechanics and Physics.
- CEA [28], GLOWINSKI-LIONS-TREMOLIERES [57], GLOWINSKI [53], [54] , [55], TREMOLIERES [81] for the numerical analysis (approximation and iterative solution).
- BENSOUSSAN-LIONS [9] for Quasi Variational Inequalities.
- BAIOCCHI-CAPELO [6], for applications to the solution of the free boundary problems related to flows in porous media (for that last subject see also BAIOCCHI-COMINCIOLI-MAGENES-POZZI [7], CRYER-FETTER, loc. cit.).

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REFERENCES

[1] BAIOCCHI C., Sur un problème à frontière libre traduisant le filtrage de liquides à travers des milieux poreux, C.R.Ac. Sc. Paris, 273 A, (1971), pp. 1215-1217.

[2] BAIOCCHI C., Su un problema di frontiera libera connesso a questioni di idraulica, Ann. Mat. Pura Appl., 4, (1972), 92, pp. 107-127.

[3] BAIOCCHI C., Free boundary problems in the theory of fluid flow through porous media, Proc. of the Int. Congress of Math. (Vancouver 1974), Vol. 2, Vancouver, (1975), pp 237-243.

[4] BAIOCCHI C., Estimations d'erreur dans L^∞ pour les inéquations à obstacle, in Mathematical Aspects of Finite Element Methods, Rome 1975, I. Galligani and E. Magenes Ed., Lecture Notes in Math., Vol. 606, Springer-Verlag, Berlin, 1977, pp. 27-34.

[5] BAIOCCHI C., BREZZI F., COMINCIOLI V., Free boundary problems in fluid flow through porous media, Proceeding of the 2nd International Symposium on Finite Element Methods in Flow Problems, S. Margherita Ligure (Italy), June 1976, ICCAD, 1976, pp. 407-420.

[6] BAIOCCHI C., CAPELO A., Disequazioni variazionali e quasivariazionali. Applicazioni a problemi di frontiera libre, (2 Vol.), Quaderni 4 e 7 dell' Unione Mathematica Italiana, Pitagora Editrice, Bologna, 1978.

[7] BAIOCCHI C., COMINCIOLI V., MAGENES E., POZZI G.A., Fluid flow through porous media ; a new theoretical and numerical approach, Publicazioni 69, L.A.N.-C.N.R., Pavia, 1974.

[8] BEGIS D., GLOWINSKI R., Chapter 7 of Résolution Numérique de Problèmes aux Limites par des Méthodes de Lagrangiens augmentés, M. Fortin and R. Glowinski Ed. (to appear).

[9] BENSOUSSAN A., LIONS J.L., Temps d'arrêt optimaux et contrôle impulsionnel, (to appear).

[10] BOURGAT J.F., DUVAUT G., Numerical Analysis of flow with or without wake past a symmetric two-dimensional profile with or without incidence, Int. J. Num. Meth. Eng., 11, (1977), pp. 975-993.

[11] BREZIS H., A new method in the study of subsonic flows, in Partial Differential Equations and related topics, J. Goldstein Ed., Lecture Notes in Math., Vol. 446, Springer-Verlag, Berlin, 1975, pp. 50-64.

[12] BREZIS H., Nouveaux théorèmes de régularité pour les problèmes unilatéraux Proc. Joint Meeting Theoretical Physicists and Mathematicians, Strasbourg 12, 1971.

[13] BREZIS H., Seuil de régularité pour certains problèmes unilatéraux, C.R. Acad. Sci. Paris, 273A, (1971), pp. 35-37.

[14] BREZIS H., Problèmes unilatéraux, J. de Math. Pures et Appliquées, IX, Série 72, (1971), pp. 1-168.

[15] BREZIS H., Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. In Contributions to Nonlinear Functional Analysis, E. Zaratonello Ed., Acad. Press, New-York, 1971, pp. 101-156.

[16] BREZIS H., Multiplicateur de Lagrange en torsion élasto-plastique, Arch. Rat. Mech. Anal., 49, (1972), pp. 32-40.

[17] BREZIS H., STAMPACCHIA G., Une nouvelle méthode pour l'étude d'écoulements stationnaires, C.R. Acad. Sci. Paris, 276 A, (1973), pp. 129-132.

[18] BREZIS H., STAMPACCHIA G., The hodograph method in fluid dynamics in the light of variational inequalities, Arch. Rat. Mech. Anal., 61, (1976), 1, pp. 1-18.

[19] BREZIS H., STAMPACCHIA G., Sur la régularité de la solution d'inéquations elliptiques, Bull. Soc. Math. France, 96, (1968), pp. 153-180.

[20] BREZIS H., SIBONY M., Equivalence de deux inéquations variationnelles et applications, Arch. Rat. Mech. Anal., 41, (1971), pp. 254-265.

[21] BREZZI F., HAGER W.W., RAVIART P.A., Error estimates for the finite element solution of variational inequalities, Part I - Primal Theory, Numer. Math., 28, (1977), pp. 431-443.

[22] BREZZI F., HAGER W.W., RAVIART P.A., Error estimates for the finite element solution of variational inequalities, Part II - Mixed Methods, Numer. Math. (to appear).

[23] BREZZI F., JOHNSON C., MERCIER B., Analysis of a Mixed Finite Element Method for Elasto-Plastic Plates, Math. of Comp., 31, (1977), 140, pp. 809-817.

[24] BRISTEAU M.O., Application of optimal control theory to transonic flow computations by finite element methods, in Proceedings of the Third IRIA Symposium on Computing Methods in Applied Sciences and Engineering, Versailles, France, Dec. 1977.

[25] BRISTEAU M.O., Application of a finite element method to transonic flow problems using an optimal control approach, V.K.I. Lecture Series : Computational Fluid Dynamics, Von Karman Institute for Fluid Dynamics, Rhode-St-Genèse, Belgium, March 1978.

[26] BRISTEAU M.O., GLOWINSKI R., PERIAUX J., PERRIER P., PIRONNEAU O., POIRIER G., Application of Optimal Control and Finite Element Methods to the calculation of Transonic Flows and Incompressible Viscous Flows, LABORIA Research Report N° 294, April 1978.

[27] CAPRIZ G., Variational Techniques for the Analysis of a lubrication problem, in Mathematical Aspects of Finite Element Methods, Rome 1975, I. Galligani and E. Magenes Ed., Lecture Notes in Math., Vol. 606, Springer-Verlag, Berlin, 1977, pp. 47-55.

[28] CEA J., Optimization : Théorie et Algorithmes, Dunod, Paris, 1970.

[29] CEA J., GLOWINSKI R., Sur des méthodes d'optimisation par relaxation, Rev. Française Automat. Informat. Rech. Opérationnelle, R-3, Dec. 1973, pp. 5-32.

[30] CEA J., GLOWINSKI R., Méthodes numériques pour l'écoulement laminaire d'un fluide rigide visco-plastique incompressible, Int. J. of Comp. Math., B, Vol. 3, (1972), pp. 225-255.

[31] CEA J., GLOWINSKI R., NEDELEC J.C., Application des méthodes d'optimisation, de différences et d'éléments finis à l'analyse numérique de la torsion élasto-plastique d'une barre cylindrique, in Approximations et méthodes itératives de résolution d'inéquations variationnelles et de problèmes non linéaires. Cahier de l'IRIA, N° 12, (1974), pp. 7-138.

[32] CHAN T.F., GLOWINSKI R., Numerical methods for solving some mildly nonlinear elliptic partial differential equations, Stanford Report, (to appear).

[33] CIARLET P.G., The finite element method for elliptic problems, North-Holland, 1978.

[34] CIAVALDINI J.F., POGU M., TOURNEMINE, Etude d'écoulements subcritiques compressibles dans le plan physique, Cas des écoulements non portants, Journal de Mécanique (to appear).

[35] COMINCIOLI V., On some oblique derivative problems arising in the fluid flow in porous media. A theoretical and numerical approach, Applied Math. Optim., 1, (1975), 4, pp. 313-336.

[36] COMINCIOLI V., Metodi di pilassamento per la minimizzazione in uno spazio prodatto, L.A.N.-C.N.R., 20, (1971), Pavia.

[37] CONCUS P., GOLUB G.H., O'LEARY D., Numerical solution of nonlinear partial differential equations by a generalized conjugate gradient method,Computing, 19, (1977), 4, pp. 321-340.

[38] COTTLE R.W., Computational experience with large scale linear complementarity problems, in Fixed Points : Algorithms and Applications, S. Karamardian Ed., Acad. Press, New-York, 1977, pp. 281-313.

[39] COTTLE R.W., Numerical Methods for Complementarity Probmems in Engineering and Applied Science, to appear in the Proceedings of the Third Iria Symposium on Computing Methods in Applied Sciences and Engineering.

[40] COTTLE R.W., GOLUB G.H., SACHER R.S., On the solution of large, structured, linear complementarity problems, Stanford Report STAN-CS-74-439, August 1974, Computer Science Department, Stanford University.

[41] CRYER C.W., The method of Christoferson for solving free boundary problems for infinite journal bearings by means of finite differences, Math. Comp., 25, (1971), pp. 435-443.

[42] CRYER C.W., The solution of a quadratic programing problem using systematic over-relaxation, SIAM J. of Control, Vol. 9, N° 3, (1971), pp. 385-392.

[43] CRYER C.W., FETTER H., The numerical solution of axisymmetric free boundary porous flow well problems using variational inequalities, MRC Technical Summary Report # 1761, University of Wisconsin, 1977.

[44] DOUGLAS J., DUPONT T., Preconditioned conjugate gradient iteration applied to Galerkin methods for a mildly nonlinear Dirichlet problem, in Sparse Matrix Computations, J.R. Bunch, D.J. Rose Ed., Academic Press, New-York, 1976, pp. 333-348.

[45] DUVAUT G., LIONS J.L., Les Inéquations en Mécanique et en Physique, Dunod, Paris, 1972.

[46] FALK R.S.. Approximate solutions of some variational inequalities with order of convergence estimates, Ph. D. Thesis, Cornell University, 1971.

[47] FALK R.S., Error estimates for the approximation of a class of variational inequalities, Math. of Comp., Vol. 28, (1974), pp. 963-971.

[48] FALK R.S., Approximation of an elliptic boundary value problem with unilateral constraints, Rev. Française Automat. Informat. Rech. Opérationnelle, R2, (1975), p. 5-12.

[49] FALK R.S., MERCIER B., Error estimates for elasto-plastic problems, Rev. Française Automat. Informat. Rech. Opérationnelle, 11, (1977), 2, pp. 135-144.

[50] FORTIN M., Minimization of Some Non-Differentiable Functionals by the Augmented Lagrangian Method of Hestenes and Powell, Applied Math. Optimization, Vol. 2, (1975/76), N° 3, pp. 236-250.

[51] FORTIN M., GLOWINSKI R. (Ed.), Résolution numérique de problèmes aux limites par des méthodes de lagrangiens augmentés, (to appear).

[52] GABAY D., MERCIER B., A dual algorithm for the solution of nonlinear variational problems via finite element approximation. Comp. and Math. with Applic., Vol. 2, (1976), pp. 17-40.

[53] GLOWINSKI R., Introduction to the approximation of elliptic variational inequalities, Report 76006, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 1976.

[54] GLOWINSKI R., Sur l'approximation d'une inéquation variationnelle elliptique de type Bingham, Rev. Française Automat. Informat. Rech. Opérationnelle, 10, (1976), 12, pp. 13-30.

[55] GLOWINSKI R., Numerical Analysis of Nonlinear Variational Problems, Lecture Notes, Tata Institute, Bombay and Bangalore (to appear).

[56] GLOWINSKI R., LANCHON H., Torsion élasto-plastique d'une barre cylindrique de section multiconnexe, J. de Mécanique, Vol. 12, (1973), 1, pp. 151-171.

[57] GLOWINSKI R., LIONS J.L., TREMOLIERES R., Analyse Numérique des Inéquations Variationnelles, (2 Vol.), Dunod - Bordas, Paris, 1976.

[58] GLOWINSKI R., MARROCCO A., Chapter 5 of Résolution Numérique de problèmes aux limites par des méthodes de lagrangiens augmentés, M. Fortin, R. Glowinski Ed., (to appear).

[59] GLOWINSKI R., PIRONNEAU O., On the computation of transonic flows, in Proceedings of the 1st Franco-Japonese Colloquium of Functional Analysis and Numerical Analysis, Tokyo, Kyoto, September 1976.

[60] GLOWINSKI R., PIRONNEAU O., Least square solution of nonlinear problems in fluid dynamics, in Comtemporary Developments in Continuum Mechanics and P.D.E., G.M. de la Penha and L.A. Medeiros Ed., North-Holland,(to appear).

[61] HLAVACHEK I., Dual finite element analysis for unilateral boundary value problems, Applikace Matematiky, 22, (1977), pp. 14-51.

[62] JOHNSON C., A mixed finite element method for plasticity problems with hardening, SIAM J. Numerical Analysis, 14, (1977), 4, pp. 575-583.

[63] JOHNSON C., An elasto-plastic contact problem, Rev. Française Automat. Inform. Rech. Opérationnelle, Numerical Analysis, 12, (1978), 1, pp. 59-74.

[64] LANDAU L., LIFCHITZ E., Mécanique des Fluides, Mir, Moscou, 1953.

[65] LIONS J.L., Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier-Villars, Paris, 1969.

[66] LIONS J.L., G. STAMPACCHIA, Variational Inequalities, Comm. Pure Applied Math., XX, (1967), pp. 493-519.

[67] MARZULLI P., Rizoluzione alle differenze finite di equazioni alle derivate parzialli di tipo ellittico con condizioni su un conterno libero, Calcolo Suppl., 1, (1968), 5, pp. 1-22.

[68] MEIJERINK J.A., VAN der VORST H.A., An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix, Math. of Comp., 31, (1977), pp. 148-162.

[69] MERCIER B., Sur la Théorie et l'Analyse Numérique de Problèmes de Plasticité, Thesis, Université de Paris VI, 1977.

[70] MOSCO U., Error estimates for some variational inequalities, in Math. Aspects of Finite Element Methods, I. Galligani and E. Magenes Ed., Lecture Notes in Math., Vol. 606, Springer-Verlag, Berlin, (1977), pp. 224-236.

[71] MOSCO U., SCARPINI F., Complementarity systems and approximation of variational inequalities, Rev. Française Automat. Info. Rech. Opérationnelle, R-1, Avril 1975, pp. 83-104.

[72] MOSCO U., STRANG G., One sided approximation and variational inequalities, Bull. AMS, 80, (1974), pp. 308-312.

[73] MOSSOLOV P.P., MIASNIKOV V.P., Variational Methods in the theory of a viscous-plastic medium, J. of Mech. and Applied Math. (P.M.M.), 29, (1965), 3, pp. 468-492.

[74] MOSSOLOV P.P., MIASNIKOV V.P., On stagnant flow regions of a viscous plastic medium in pipes, J. of Mech. and Applied Math. (P.M.M.), 30, (1966), 4, pp. 707-717.

[75] MOSSOLOV P.P., MIASNIKOV V.P., On qualitative singularities of the flow of a viscous plastic medium in pipes, J. of Mech. and Applied Math. (P.M.M.), 31, (1967), 3, pp. 581-585.

[76] NATTERER F., Optimale L_2-Konvergenz finiten elemente bei variationsung lei chungen, (to appear).

[77] NITSCHE J., L_∞-convergence of finite element approximations, in Math. Aspects of Finite Element Methods, I. Galligani and E. Magenes Ed., Lecture Notes in Math., Vol. 606, Springer-Verlag, Berlin, 1977, pp. 261-274.

[78] ROUX J., Résolution numérique d'un problème d'écoulement subsonique de fluide compressibles, Rev. Française Automatique Informatique Recherche Opérationnelle, Sér. Rouge, Analyse Numérique, 10, (1976), 12, pp. 31-50.

[79] SCARPINI F., VIVALDI M.A., Error estimates for the approximation of some unilateral problems, Rev. Française Automat. Informat. Rech. Opérationnelle, Anal. Num., 11, (1977), pp. 197-208.

[80] SHAW F.S., The torsion of solid and hollow prisms in the elastic and plastic range by relaxation methods, Report ACA-11, 1944.

[81] TREMOLIERES R., Inéquations Variationnelles : Existence, Approximation, Résolution. Thesis, Université de Paris VI, 1972.

[82] VARGA R.S., Matrix iterative analysis, Prentice-Hall, 1962.

[83] WHITEMAN J.R., NOOR M.A., Finite element errors for nonlinear variational inequalities, (to appear).

[84] YOUNG D.M., Iterative solution of large linear systems, Acad. Press, New-York, 1971.

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In this report we consider the various problems related to the numerical analysis of elliptic variational inequalities. After proving some abstract results concerning the approximation of variational inequalities, we consider the approximation of some specific examples by conforming and non conforming finite element methods. Various iterative methods of solution are also described.

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